Some Two-Dimensional Inclusion Problems And Point Force Problems In Elasticity

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SPEECH CODING BY COMPLEX AM AND FM SIGNAL MODELS

By

Deshraj Singh

DEPARTMENT OF ELECTRICAL ENGINEERING

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CERTIFICATE

This is to certify that the thesis entitled 'Some Two-Dimensional Inclusion Problems And Point Force Problems in Elasticity ' that is being submitted by Shri S. C. Gupta, M.Sc. for the award of the Degree of Doctor of Philosophy to the Indian Institute of Technology, Kanpur is a record of bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standard fulfilling the requirements of the regulations to the Degree. The results embodied in this thesis have not been submitted to any other university or Institute for the award of any degree or diploma.

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I acknowledge with gratitude the valuable advice and guidance of my supervisor, Professor R. D. Bhargava, who introduced me to the subject and suggested me the problems.

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SHORT LIST OF SYMBOLS

(x,y)	Cartesian coordinates of a point in two dimensions
(r,θ)	Polar coordinates of a point
(ξ,η)	Elliptic coordinates of a point
(x,y,z)	Cartesian coordinates of a point in three dimensions
(R,θ,Z)	Cylindrical polar coordinates of a point
(r,θ,φ)	Spherical polar coordinates of a point
(u,v)	Displacement components in two dimensional Cartesian coordinate system
(u_r,u_θ)	Displacement components in polar coordinate system
(u_x,u_y,u_z)	Displacement components in three dimensional Cartesian coordinate system
(u_R,u_θ,u_Z)	Displacement components in cylindrical polar coordinates
(u_r,u_θ,u_φ)	Displacement components in spherical polar coordinates
P_{xx},P_{yy},P_{xy}	Stresses in two-dimensional Cartesian coordinate system
$P_{rr},P_{\theta\theta},P_{r\theta}$	Stresses in polar coordinates
$P_{RR},P_{\theta\theta},P_{ZZ}$, $P_{r\theta},P_{rZ},P_{\theta Z}$	Stresses in cylindrical coordinates
$P_{rr},P_{\theta\theta},P_{\varphi\varphi}$, $P_{r\theta},P_{r\varphi},P_{\theta\varphi}$,	Stresses in spherical polar coordinates
λ, μ	Lamé constants
E	Young's modulus
ν	Poisson ratio

SYNOPSIS

In this thesis, some elastic inclusion problems in two-dimensional regions and point force problems in three-dimensional regions in elasticity theory have been studied. The thesis is divided into two parts. Part A is concerned with some two-dimensional inclusion problems. Part B deals with point force problems.

The inclusion is a region in an elastic medium (called the matrix). This region (inclusion) would undergo a prescribed deformation in the absence of the surrounding material (the matrix). But because of the constraints of the matrix, stresses will develop everywhere in the material. The problem is to find the elastic field in the inclusion and the matrix.

Part A deals with such problems. In the works of the previous authors, the inclusions were supposed to be present in an infinite or semi-infinite medium. The methods of solutions of such problems were based upon 'point force' technique or energy considerations. If the matrix is finite these methods do not succeed because point force requires an infinite or semi-infinite medium and energy considerations require considerable guessing of the equilibrium interface.

Practically no problems were earlier solved for finite regions. Fortunately, however, when the matrix is finite, the theory of Hilbert problem and complex variable method can be applied. As far as is known to the author, this method has been applied for the first time in this thesis after suitable adaptation. Exact analytical solutions have been obtained for some problems. These problems are (i) an eccentric circular inclusion in a circular region, (ii) an elliptic inclusion in a circular region and (iii) rectangular inclusion in a circular region. The solutions can be adopted to the case when the Poisson ratios of the inclusion and matrix are different but shear moduli the same. It may be noted that the results obtained by other authors for infinite medium can be obtained as particular cases of the solutions obtained in this thesis by suitable limiting processes. This has been indicated in this thesis.

As stated above, the point force technique has proved to be quite useful when inclusions are present in an infinite medium. However, it may be mentioned that the point force is not only useful for inclusion problems but even otherwise it has found applications in material testing, soil mechanics and engineering structures. Thus the results for point force are interesting in themselves. Hence in the Part B of the

thesis , we have solved some point force (also called concentrated force) problems.

The first problem is that of a cone under an axial point force acting at a point on the axis other than the vertex of the cone. It may be mentioned that the problem of axial point force acting at the vertex of a cone was solved by J.H. Michell in 1900 and since then no progress was made. The results of our problem involve infinite integrals which are convergent. To have the feeling for the stresses in the cone for this case, the integrals were evaluated numerically for some cases and have been given in the thesis at suitable places. When the point force acts very near the vertex, it is obvious on physical grounds that the stresses cannot differ significantly from the case when the point force acts at the vertex. Numerical results confirm this expectation and are reported in the thesis.

The second problem is that of an infinite cylinder. Here again the axial point force acts at a point on the axis of the cylinder, which has been chosen as the origin. The method is essentially the same as that employed for the cone. The results involve infinite integrals again. These are evaluated numerically and presented in the thesis. It may be noted that when the radius of the cylinder is very large , we deduce the case of a point force in an infinite medium.

The next problem is that of a point force in a wedge. The point force acts at any point (not necessarily on the axis of the wedge) and is inclined to the axis of the wedge at any angle. The results are tabulated for some particular cases.

In all the cases of point force, an auxiliary problem when tractions are prescribed on the surfaces in the case of cone and cylinder and on the faces in the case of wedge, is also solved.

Lastly the problem of a circular inclusion in a wedge is solved. Results are given for a symmetrically situated inclusion. The problem of a circular inclusion in a half plane and an infinite medium can be deduced as particular cases.

INTRODUCTION

As stated in the synopsis, this thesis deals with some inclusion problems in two-dimensions and point force problems in three-dimensions in the classical isotropic elasticity theory. The definitions and the problems are also stated in the synopsis.

The study of inclusion problems was first initiated by Frenkel ((25)) in connection with his kinetic theory of liquids and by Mott and Nabarro ((26)) in connection with their theory of precipitation hardening of alloys . Further progress was made only after Eshelby's papers ((11, 14)) dealing with the elastic field of an ellipsoidal inclusion.

Since then rapid progress has been made. Reference may be made to the classical work of Jaswon and Bhargava ((15)), where for two-dimensional problems the point force was coupled with complex variable technique to obtain explicit solutions for elliptic inclusion problem. The technique could be further applied to anistropic bodies and this was done by Willis in ((27)) and by Bhargava and Kapoor ((28, 9)) to obtain the solution for the triangular and rectangular inclusions in an infinite medium. The

latter authors also solved some more interesting problems relating to inclusions in half plane and the problems where the inclusion interacted with another inclusion or an inhomogeneity or a cavity in an otherwise infinite medium. Sharma ((29)) considered the circular inclusion in an infinite strip.

In 1961, Bhargava ((10)) showed that the principle of minimum strain energy can also be successfully employed and is more useful when in place of inclusion there is an inhomogeneity. This idea was used by Bhargava and Radhakrishna to obtain some explicit solutions for isotropic ((12)) and anistropic ((13)) elliptic inhomogeneities in an infinite medium. Dundurs and Mura ((30)) have investigate the behaviour of an edge dislocation present in the matrix with a circular inclusion. Dundurs and Sendeckyj ((31)) have studied a similar problem in which the dislocation is present inside the circular inclusion. Applications to some of the results to iron nitride precipitates in iron were made by Bhargava and Mclean ((32)).

Knops ((33)) derived an equation for the strains of an arbitrary elastic field in an infinite matrix perturbed by several inclusions and solved the problem when the shear moduli of inhomogeneities and matrix are identical. Some other recent contributions in this field

of study have been concerned with using variational methods to derive bounds for the aggregate moduli of multiphased materials having arbitrary phase geometry. Hill ((34)) estimated the overall moduli of an arbitrary fibre composite with transversely isotropic phases and also the macroscopic elastic moduli of two phase composites ((35)). Budiansky ((36)) gave an analysis for the determination of the elastic moduli of a composite material. The bounds for elastic moduli of solid composite materials were given by Walpole ((37, 38)) by employing extremum principles.

In all the problems mentioned above, the matrix was taken as infinite or semi-infinite. As far as is known to the author, inclusion problems when the matrix is finite, has been solved only in this thesis. Such problems are more realistic and technologically important.

As mentioned earlier, the explicit solutions were obtained either by point force technique or energy considerations. The point force requires an infinite or semi-infinite media and energy considerations require considerable guessing of the equilibrium interface, both of which are not easily possible if the matrix is finite. However, the theory of Hilbert problem coupled with complex variable can be applied to solve some inclusion problems which are reported in Part A of the thesis.

Part A consists of four chapters. In the first chapter some results of the complex variable formulation of two-dimensional elastostatic problems which are needed in subsequent chapters are given. Also theory of Hilbert problem and the method of application of this theory to inclusion problems is briefly recounted in this chapter.

In chapter II, the problem of eccentric circular inclusion in a circular ring is solved. Exact analytical solution has been obtained. The results of concentric circular inclusion in a circular region, circular inclusion in an infinite medium and circular inclusion in a semi-infinite medium are obtained as particular cases of our problem. Variation of stresses at the equilibrium boundary both for inclusion and matrix is shown for some cases in the form of graphs which are given in Appendix to chapter II.

Chapter III deals with the problem of concentric elliptic inclusion in a circular region. The results of elliptic inclusion in an infinite medium and the results of concentric circular inclusion may be derived as particular cases of our problem. Lines of maximum shearing stress have been drawn for matrix and their graphs are given in Appendix to chapter III.

The problem of rectangular inclusion in a circular region is solved in chapter IV. An exact analytical

solution has been given. Square inclusions in a circular and infinite region and rectangular inclusion in an infinite region are obtained as particular cases. For matrix lines of maximum shearing stress have been drawn and are given in the form of graphs in Appendix to chapter IV. It is interesting to note that the lines of maximum shearing stress emanating from the boundaries of slender ellipses resemble to some extent with those emanating from the boundaries of rectangles with small width.

As stated earlier point force technique has been widely used by some authors in solving inclusion problems. The results for point force are thus useful for inclusion problems. But elastic field due to a point force is important by itself also and has useful applications in soil mechanics, material testing and engineering structures etc.

The solution of the problem of a point force at any point in a three-dimensional infinite elastic medium was given by Lord Kelvin ((39)) and is described in Love's book ((16)). This solution has been extended to the case of a semi-infinite elastic body under a point force by Dean, Parsons, and Sneddon ((40)) and is readily available in ((17)). Michell ((41)) gave the solution of the problem of axial point force acting at

the vertex of the cone. The complex potentials corresponding to a point force in a two-dimensional infinite elastic plane are given in Green and Zerna ((1)) and by Bhargava and Kapoor for a point force at the boundary of a circular inhomogeneity in an infinite medium ((42)) and also in a semi-infinite medium having a circular cavity ((43)).

In Part B of the thesis some point force problems in three-dimensional and two-dimensional regions have been considered.

The technique which is employed in the Part B of the thesis may be very briefly described as follows : Consider an infinite elastic region in which a point force is acting, we shall get a system of stresses everywhere which is being called the first stress system. Imagine the surface (in our cases, a cone or a cylinder or a wedge) within which the elastic field is to be considered. On the surface of the body, tractions would be acting. Nullify the tractions by applying equal and opposite tractions and get a second system of stresses. Superpose the two stress systems. This gives the required stresses within the body.

As the problems, that we shall deal with are axially symmetric, the theory and some results are briefly described in chapter V. (Details may be found in ((16)) and

((17))). In this chapter the following results are given: (i) The equations of equilibrium in cylindrical polar coordinates, the solution of which depends upon two functions. (ii) The displacement components due to a point force in three-dimensional region. (iii) The stresses in a two-dimensional region under a point force. They are needed in the subsequent chapters.

In chapter VI, we consider the case of an axial point force acting at a point on the axis of the semi-infinite elastic cone. It may be noted that the previous solution by Michell ((41)) deals with the problem of axial point force acting at the vertex of the cone. As stated earlier, first we find the stresses in a infinite medium due to the point force. We get the first system of stresses. Now in this infinite media, imagine a cone (origin is the vertex of the cone and axis of the cone coincides with the direction of the point force acting at the point (0, 0, d)) and find the tractions on its surface. We apply the tractions on the surface of the cone equal and opposite to that obtained from first stress system. This gives rise to another stress field in the cone which we call the second system of stresses. These two stress systems are superposed and we obtain the stresses in the cone due to a point force with zero tractions on its surface. The results

for the first and second system of stresses are given in this chapter.

In chapter VII, the mathematics employed for finding the second system of stresses is given, as it is not trivial. This involves the results of chapter V. First results of chapter V are converted into spherical polar coordinates. The two functions mentioned in chapter V satisfy equations (112) p. given in chapter V and when Mellin transform of these equations is taken, the solution of the transformed equations involves two unknown constants which are determined with the help of two boundary conditions. Substituting the values of the constants in the stresses, we get the second system of stresses. The stresses are given in terms of infinite integrals.

In chapter VIII, some graphs and tables for some interesting values of d (the distance of the point of application of the force from the vertex of the cone) are given. This is necessary because the results of the second system of stresses involve infinite integrals and a method is needed to evaluate the stresses numerically. As check on the numerical work, one of the values of d was taken to be 0.01 in non-dimensional form so that the point force acts

very near the vertex of the cone. It is obvious on physical grounds that the stresses cannot differ significantly from the case when the point force acts at the vertex. Numerical results confirm this expectation. As the results involve infinite integrals, it is necessary to consider the convergence of the integrals which has been done in this chapter.

In chapter IX, the problem for an axial force in an infinite elastic cylinder is considered. The method is essentially the same as described in chapter VI. Therefore only the results for the first and second stress systems are given. The second system of stresses involve infinite integrals. The convergence of the infinite integrals is discussed.

In Appendix to this chapter, some graphs are given for the stress field for various sizes of the cylinder.

In chapter X, we deal with the point force problem in a wedge in two-dimensional elasticity theory. But this force acts at any point and in any direction in the wedge. Thus we have considered a point force (x_o, y_o) acting at (h, k). The method of solution is the same as in chapter VI. In Appendix to this

chapter, the convergence of the integrals is discussed and also some graphs and tables are given. Results obtained by the techniques of the chapter VIII are compared with the known solutions in a particular case and are in very good agreement.

Lastly in chapter XI, we consider the problem of circular inclusion in an infinite elastic wedge. The results are applicable for circular inclusion anywhere within the wedge. Results of circular inclusion in a half plane may be obtained as a particular case.

The work in chapters II, III, IV, VI, VII, VIII and IX is based upon the following papers some of which are published and others are under publication.

1. Eccentric circular inclusion in a circular region (Published in the Journal of Physical Society of Japan, Vol. 25, No. 3, 1968).
2. Elliptic inclusion in a circular region (Under publication in Proceedings National Institute of Sciences, India).
3. Rectangular inclusion in a circular inclusion (Under publication in Bulletin de L' academie Polanaise des Sciences, Poland).

4. Stresses in an elastic cone due to an axial force at a point on the axis. (Published in Acta Mechanica, Austria, Vol.6, No.4, 1968).
5. Stresses in an infinite elastic cylinder due to a point force along the axis.(Published in International Journal of Engineering Science, U.S.A., Vol.7, No.7, 1969).

P A R T A

CHAPTER I

COMPLEX VARIABLE FORMULATION AND THE HILBERT PROBLEM

This chapter describes the formulation of two-dimensional elastostatic problems by the complex variable approach and the problem of linear relationship (the Hilbert problem). This is done to make the thesis self-contained.

The method of solution of two-dimensional problems in infinitesimal theory of elasticity and the history of development have been exhaustively dealt in the books of Green and Zerna ((1)), Muskhelishvili ((2)), Sokolnikoff ((3)) and Timoshenko and Goodier ((4)) *et al* .

In the two-dimensional elastostatic problems of plane strain and generalized plane stress, the equations of equilibrium in the absence of body forces are

$$\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} = 0 \quad ,$$

$$\frac{\partial P_{yx}}{\partial x} + \frac{\partial P_{yy}}{\partial y} = 0 \quad , \tag{1}$$

where P_{ij} $(i, j = x,y)$ are the stress components and x,y is the Cartesian coordinate system.

The compatibility condition which is satisfied in the region R under consideration is

$$\nabla^2(P_{xx} + P_{yy}) = 0 \quad . \tag{2}$$

The conditions on the boundary C of R are ,

$$P_{xx} \cos(n,x) + P_{yx} \cos(n,y) = P_{nx}$$

and $$P_{xy} \cos(n,x) + P_{yy} \cos(n,y) = P_{ny} \quad , \tag{3}$$

where n denotes the normal to the surface element at (x,y).

As noted by G. B. Airy, the equations of equilibrium in (1) are satisfied by substituting

$$P_{xx} = \frac{\partial^2 U}{\partial y^2} , \quad P_{xy} = -\frac{\partial^2 U}{\partial y \partial x} \quad \text{and} \quad P_{yy} = \frac{\partial^2 U}{\partial x^2} , \tag{4}$$

where $U = U(x,y)$ is a real function. Substituting (4)

in (2), leads to the biharmonic equation

$$\nabla^4 U = 0 \quad . \tag{5}$$

The solution of (5) in terms of two analytic functions $\phi(z)$ and $\chi(z)$ can be written as ((3))

$$2U = \bar{z}\,\phi(z) + z\,\overline{\phi(z)} + \chi(z) + \overline{\chi(z)} , \tag{6}$$

where $z = x + iy$, and bar stands for complex conjugate.

From (4) and (6)

$$\begin{aligned} P_{xx} + i\,P_{xy} &= \frac{\partial^2 U}{\partial y^2} - \frac{i\,\partial^2 U}{\partial y \partial x} , \\ P_{yy} - i\,P_{xy} &= \frac{\partial^2 U}{\partial x^2} - \frac{i\,\partial^2 U}{\partial y \partial x} . \end{aligned} \tag{7}$$

Also,

$$\frac{\partial U}{\partial x} + \frac{i\partial U}{\partial y} = \phi(z) + z\,\overline{\phi'(z)} + \overline{\psi(z)} , \tag{8}$$

where dash means differentiation with respect to the argument throughout the thesis and $\psi(z) = \chi'(z)$.

Finally, the stresses and displacements in Cartesian coordinate system are given by

$$\begin{aligned} P_{xx} + P_{yy} &= 4\mathrm{Re}\left[\phi'(z)\right] \\ P_{yy} - P_{xx} + 2i\,P_{xy} &= 2\left[\bar{z}\,\phi''(z) + \psi'(z)\right] \end{aligned} \tag{9}$$

and $2\mu(u + iv) = K\,\phi(z) - z\,\overline{\phi'(z)} - \overline{\psi(z)}$, (10)

where Re stands for the real part of a complex quantity,

$K = 3-4\nu$ for plane strain and $K = \frac{3-\nu}{1+\nu}$ for generalised plane stress, ν being the Poisson ratio.

The analytic functions $\phi(z)$ and $\psi(z)$ are determined uniquely for a given state of stress, if the origin of coordinates is taken within the region under consideration and the following conditions are imposed

$$\phi(0) = 0 \quad , \quad I\{\phi'(0)\} = 0 \quad \text{and} \quad \psi(0) = 0, \tag{11}$$

where I stands for the imaginary part of a complex quantity. For the first boundary value problem i.e. when the tractions are prescribed on the boundary C of R, the boundary condition is

$$\phi(z) + z\,\overline{\phi'(z)} + \overline{\psi(z)} = f_1(s) + i\,f_2(s) + \text{constant on } C, \tag{12}$$

where $f_1(s) + i\,f_2(s) = i\int^{s}(P_{nx} + i\,P_{ny})ds$.

Conformal transformations have been utilized effectively in two-dimensional elasticity problems. A brief account of the conformal mapping and the methods of determining the analytic functions $\phi(z)$ and $\psi(z)$ are given below.

Suppose the region R (finite or infinite) whose boundary C has continuously changing curvature is simply connected and there is an analytic function

$$z = w(\zeta) \quad , \quad \zeta = \rho e^{i\nu} \tag{13}$$

which maps the region R conformally on the unit circle

We distinguish a point on the boundary of $|\zeta| \leq 1$ by writing $\zeta = \sigma$.

Denoting

$$\phi_1(\zeta) = \phi\{\omega(\zeta)\}, \quad \psi_1(\zeta) = \psi\{\omega(\zeta)\} \quad \text{and} \quad \phi'(z) = \phi_1'(\zeta)\frac{1}{\omega'(\zeta)} \tag{14}$$

and noting that $\zeta = \sigma$ on the boundary of the unit circle equation (12) may be written as

$$\phi_1(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\phi_1'(\sigma)} + \overline{\psi_1(\sigma)} = H(\sigma) \tag{15}$$

where $H(\sigma) = f_1(\vartheta) + i f_2(\vartheta)$ and $\sigma = e^{i\vartheta}$ on the boundary of the unit circle.

The expressions of stresses are given by the formulas

$$P_{\rho\rho} + P_{\vartheta\vartheta} = 4\,\mathrm{Re}\,[\,\Phi(\zeta)] \tag{16}$$

$$P_{\vartheta\vartheta} - P_{\rho\rho} + 2i\,P_{\rho\vartheta} = \frac{2\zeta^2}{\rho^2\,\overline{\omega'(\zeta)}}\left[\,\overline{\omega(\zeta)}\;\Phi'(\zeta) + \omega'(\zeta)\,\Psi(\zeta)\right] \tag{17}$$

where $\Phi(\zeta) = \dfrac{\phi_1'(\zeta)}{\omega'(\zeta)}$ and $\Psi(\zeta) = \dfrac{\psi_1'(\zeta)}{\omega'(\zeta)}$.

For a finite simply connected region R, the functions $\phi_1(\zeta)$ and $\psi_1(\zeta)$ may be represented in the power series

$$\phi_1(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n \quad \text{and} \quad \psi_1(\zeta) = \sum_{n=0}^{\infty} b_n \zeta^n \quad \text{in R.} \tag{18}$$

Both $H(\sigma)$ and $\dfrac{\omega(\sigma)}{\overline{\omega'(\sigma)}}$ in (15) can be expanded by the complex Fourier series as

$$H(\sigma) = \sum_{-\infty}^{\infty} A_k \sigma^k \quad \text{and} \quad \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} = \sum_{-\infty}^{\infty} B_k \sigma^k \quad . \qquad (19)$$

By substituting these series expansions in (18) and (19) in (15), and on comparing the coefficient of different powers of σ on both sides of the equation, the undetermined coefficients a_n and b_n are obtained.

In order to obtain unique solution in the first boundary value problem $I\{\phi_1'(0)/\omega'(0)\}$ is assigned and $\phi_1(0) = 0$.

The analytic functions $\phi_1(\zeta)$ and $\psi_1(\zeta)$ can also be determined with the help of integrodifferential equations. Multiplying both sides of (15) by $\frac{1}{2\pi i}\frac{d\sigma}{\sigma-\zeta}$, where $|\zeta|<1$ and integrating over γ the boundary of the unit circle, it becomes

$$\frac{1}{2\pi i}\int_\gamma \frac{\phi_1(\sigma)}{\sigma-\zeta}\,d\sigma + \frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\phi_1'(\sigma)}}{\sigma-\zeta}\,d\sigma + \frac{1}{2\pi i}\int_\gamma \frac{\overline{\psi_1(\sigma)}}{\sigma-\zeta}\,d\sigma = A(\zeta) \qquad (20)$$

where $$A(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{H(\sigma)\,d\sigma}{\sigma-\zeta} \quad .$$

By Harnack's theorem, equation (15) and (20) are equivalent.

The analyticity of $\phi_1(\zeta)$ and $\psi_1(\zeta)$ reduces (20) to the desired integrodifferential equation

$$\phi_1(\zeta) + \frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{\phi_1'(\sigma)}}{\sigma-\zeta}\,d\sigma + \overline{\psi(0)} = A(\zeta) \quad . \qquad (21)$$

The value of the unknown constant $\overline{\psi(0)}$ may be determined by imposing the condition $\phi_1(0)=0$.

$\psi_1(\zeta)$ may be determined by multiplying the conjugate

of (15) by $\frac{1}{2\pi i} \frac{d\sigma}{\sigma - \zeta}$ and integrating over γ . The required equation is

$$\psi_1(\zeta) = \frac{1}{2\pi i} \int_\gamma \frac{\overline{H(\sigma)}}{\sigma - \zeta} d\sigma - \frac{1}{2\pi i} \int_\gamma \frac{\overline{w(\sigma)}}{w'(\sigma)} \frac{\phi_1'(\sigma)}{\sigma - \zeta} d\sigma \; . \qquad (22)$$

This above formulation of the two-dimensional elastostatic problems in terms of the analytic functions $\phi(z)$ and $\psi(z)$ shall suffice for the purpose of this thesis.

Theory of Hilbert problem is now stated below in brief. This theory is described in details in ((1)). For defining the Hilbert problem, we begin with the definition of a line.

A line is defined as the union of a finite number of simple, smooth, non-intersecting arcs and contours in the complex z plane. The line is denoted by L.

It is assumed that each arc or contour which is a component of L has a definite positive direction. The ends of the arcs, if such exist, form part of L and are called ends of the line L. These ends are denoted by $a_k b_k$, $k = 1,2,\ldots$.

Draw a circle of sufficiently small radius about any point t which does not coincide with one of the ends of L. The two parts into which the circle is divided by the line L are called the left and right neighbourhoods of the point t on L according as they lie on the left or right of the line on looking in the positive direction of L. In a similar manner the left and right neighbourhoods of any

part of the line L may be defined. These left and right neighbourhoods are distinguished by (+) and (-) respectively.

A function F(z) is continuous from the left or right at a point t on L, if F(z) tends to a limit as z tends to t along any path remaining however on the left or right of L. The limiting values are denoted by

$$F^{+}(t) \text{ or } F^{-}(t) \quad ..$$

Let S' be the z plane cut along L and F(z) be some function, given in S' but not on L and satisfying the following conditions

1. The function F(z) is holomorphic everywhere in S'.
2. F(z) is continuous from the left and from the right at all points of L, other than the ends $a_k b_k$.
3. Near the ends $a_k b_k$

$$|F(z)| < \frac{A}{|z-c|^{1}}, \quad 0 \leqslant 1 < 1, \tag{23}$$

where c is any one of the ends $a_k b_k$, A and 1 are constants. Then such a function F(z) is called sectionally holomorphic in the entire z-plane or simply sectionally holomorphic. The line L is called the line of discontinuity of F(z).

Now the Hilbert problem is to find the sectionally holomorphic function F(z) with the line of discontinuity L, the boundary values of which from the left and from the right, satisfy the condition

$$F^{+}(t) = G(t)\, F^{-}(t) + f(t) \text{ on } L , \tag{24}$$

except at the ends. $G(t)$ and $f(t)$ are the functions given on L and $G(t) \neq 0$ everywhere on L. $G(t)$ and $f(t)$ satisfy the conditions in (23).

The simplest case of the problem occurs when $G(t) = 1$. The equation (24), now becomes

$$F^{+}(t) - F^{-}(t) = f(t) \quad \text{on} \quad L . \tag{25}$$

The solution of (25), may be written as a Cauchy integral

$$F_{*}(z) = \frac{1}{2\pi i} \int_{L} \frac{f(t)dt}{t-z} . \tag{26}$$

It may be seen from (26) that $F_{*}(z)$ vanishes at infinity and satisfies (25).

For finite regions, the solution of (25) may be given with the help of the following result.

If $F(z)$ is sectionally holomorphic in some finite region S_o, then this function $F(z)$ may always be represented in the form of the sum of a function $F_o(z)$ holomorphic in S_o and a Cauchy integral $F_{*}(z) = \frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z}$. Thus $F(z) = F_o(z) + F_{*}(z)$ or more explicitly

$$F(z) = F_o(z) + \frac{1}{2\pi i} \int_{L} \frac{f(t)dt}{t-z} , \tag{27}$$

L is the line of discontinuity and $f(t)$ is given by (25).

The expression in (27) hold true everywhere in S_o, except at the points of L.

The theory given above may be used to solve some two-dimensional elastostatic problems as follows. Let S_o be the two-dimensional region under consideration. The outer boundary of S_o is denoted by L_o which for the purpose of the subsequent chapters, may be assumed as simple closed contour. The discontinuities in the displacement components for a passage through L are given by

$$u^+(t) - u^-(t) = g_1(t) \quad ,$$
$$v^+(t) - v^-(t) = g_2(t) \quad \text{on } L \, , \tag{28}$$

where $g_1(t)$ and $g_2(t)$ are known functions and (+) and (-) superscripts are the same as defined previously.

If X_n and Y_n are Cartesian components of the external stress vector acting on the boundary L_o, and n is the outward normal, then the following boundary conditions are satisfied.

$$\phi(t) + t\,\overline{\phi'(t)} + \overline{\psi(t)} = f(t) \text{ on } L_o \, , \tag{29}$$

$$\phi^+(t) + t\,\overline{\phi'^+(t)} + \overline{\psi^+(t)} = \phi^-(t) + t\,\overline{\phi'^-(t)} + \overline{\psi^-(t)} \quad \text{on } L, \tag{30}$$

$$K\,\phi^+(t) - t\,\overline{\phi'^+(t)} - \overline{\psi^+(t)} = K\,\phi^-(t) - t\,\overline{\phi'^-(t)}$$

$$- \overline{\psi^-(t)} + 2\mu\,g(t) \quad \text{on } L, \tag{31}$$

where $\quad f(t) \quad = \quad i \int_0^s (X_n + i\,Y_n)\, ds$ on L_o (32)

and $\quad g(t) \quad = \quad g_1(t) + i\, g_2(t)$ on L (33)

are known functions and $\phi(z)$ and $\psi(z)$ are the same as in (10) and (12).

From (29), (30) and (31), we obtain

$$\phi^+(t) - \phi^-(t) = \frac{2\mu\,g(t)}{K+1} \quad \text{on } L \tag{34}$$

and

$$\psi^+(t) - \psi^-(t) = \frac{2\mu\,h(t)}{K+1} \quad \text{on } L, \tag{35}$$

where

$$h(t) = -\,\overline{g(t)} - \bar{t}\,g'(t)\;, \quad g'(t) = \frac{dg(t)}{dt}\;. \tag{36}$$

The solutions of (34) and (35) as given in (27) may now be written as

$$\phi(z) = \phi_o(z) + \phi_*(z) \tag{37}$$

and

$$\psi(z) = \psi_o(z) + \psi_*(z) \tag{38}$$

where $\phi_o(z)$ and $\psi_o(z)$ are functions holomorphic in S_o and

$$\phi_*(z) = \frac{\mu}{\pi i(K+1)} \int_L \frac{g(t)\,dt}{t - z} \tag{39}$$

and $$\psi_*(z) = \frac{\mu}{\pi i(K+1)} \int_L \frac{h(t)dt}{t-z} . \qquad (40)$$

It may be noted that the functions $\phi(z)$ and $\psi(z)$ replace $F(z)$ in (27), and similarly $\phi_o(z)$, $\psi_o(z)$ replace $F_o(z)$ and $\phi_*(z)$ and $\psi_*(z)$ replace $F_*(z)$.

The integrals in (39) and (40) have two values depending upon whether z lies inside L or outside L. Substituting (39) and (40) in the boundary condition (29), we obtain

$$\phi_o(t) + t\,\overline{\phi_o'(t)} + \overline{\psi_o(t)} = f_o(t) \text{ on } L_o \qquad (41)$$

where $$f_o(t) = f(t) - \phi_*(t) - t\,\overline{\phi_*'(t)} - \overline{\psi_*(t)}$$

and $f_o(t)$ is a known function on L_o .

The functions $\phi_o(z)$ and $\psi_o(z)$ may be found out with the help of series expansion method or integrodifferential equations as described earlier.

The results of this chapter are applied to a few technically important problems in the following three chapters.

CHAPTER II

ECCENTRIC CIRCULAR INCLUSION IN A CIRCULAR REGION

The problem of eccentric circular inclusion in a circular region is considered in this chapter. The problem may be stated as follows :

A circular ring of outer radius R has an eccentric circular hole. The outer boundary of the circular ring is denoted by L_o and its inner boundary by L . The centre of L_o is denoted by O' and that of L by O. The radius of L is r' and c is the distance between O and O' (Figure 1 p. 48). This medium of circular ring is called matrix. If an elastic solid called 'inclusion' of dimensions slightly bigger than those of the hole, but remaining within the limits of proportional elasticity is embedded in the matrix, then because of the misfit in size, stresses would develop both in the matrix and in the inclusion .

The problem can be visualized as follows :

Consider a circular region (called inclusion), eccentrically situated in a circular ring (called matrix). This inclusion undergoes a spontaneous stress free transformation which in the absence of the matrix is prescribed. But because of constraints of the matrix, the stresses and strains appear both in the matrix and inclusion. The problem is to evaluate this elastic field everywhere. It may be seen that the mathematical solution is the same for the two problems stated in this and previous paragraphs.

The inclusion in the absence of the surrounding material undergoes a displacement whose components are characterised by $(\epsilon_1 x + \delta_1 y,\ \epsilon_2 y + \delta_2 x)$ with respect to the origin at O i.e. the centre of L .

Let (u^+, v^+) be the displacement components of the inclusion and (u^-, v^-) be those of the matrix.

At the equilibrium interface, if the displacement components are denoted by (u_b^+ , v_b^+) and (u_b^- , v_b^-), then

$$
\begin{aligned}
u_b^+ - u_b^- &= -\ (\epsilon_1 x + \delta_1 y) = g_1(t) , \\
v_b^+ - v_b^- &= -\ (\epsilon_2 y + \delta_2 x) = g_2(t) .
\end{aligned}
\tag{42}
$$

It may be remarked that (u_b^+, v_b^+) , the displacement at the boundary of the inclusion is measured from its

natural state; the state it would have been in the absence of the matrix; while (u_b^- , v_b^-) is the displacement at the boundary of matrix measured from its natural state, which is the state when the inner hole is circular.

The method of solution of this problem is based upon Hilbert's theorem for finite regions which has been stated in chapter I .

Using complex variable approach and Hilbert theorem, the problem is reduced to the determination of two sectionally holomorphic functions $\phi(z)$ and $\psi(z)$, having the line of discontinuity L. Each $\phi(z)$ and $\psi(z)$ may be expressed as the sum of a function analytic in the whole region i.e. $|z-c| \leqslant R$ and Cauchy integrals evaluated along the circle L . First $g(t)$ and $h(t)$ are evaluated from (33) and (36) and then $\phi_*(z)$ and $\psi_*(z)$ from (39) and (40).

The equation of L is $|z| = r'$. For a point on L, z $(z = x + iy)$ is denoted by t. Substituting the values of $g_1(t)$ and $g_2(t)$ from (42) in (33) and (36), we get

$$g(t) = -\left\{(\epsilon_1+\epsilon_2) - i(\delta_1-\delta_2)\right\}\frac{t}{2} - \left\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\right\}\frac{r'^2}{2t}, \qquad \ldots (43)$$

$$h(t) = (\epsilon_1 + \epsilon_2)\,\bar{t} + \{\epsilon_1 - \epsilon_2 - i(\delta_1 + \delta_2)\}\,\frac{t}{2} -$$

$$- \{\epsilon_1 - \epsilon_2 + i(\delta_1 + \delta_2)\}\, r'^2 \frac{\bar{t}}{2t^2}. \quad (44)$$

These values of $g(t)$ and $h(t)$ are substituted in (39) and (40) and after evaluating contour integrals, we get

$$\phi_{*i}(z) = -\frac{\mu\{\epsilon_1 + \epsilon_2 - i(\delta_1 - \delta_2)\}}{(K + 1)}\, z\,,$$

$$\psi_{*i}(z) = \frac{\mu\{\epsilon_1 - \epsilon_2 - i(\delta_1 + \delta_2)\}\, z}{(K + 1)}\,, \quad (45)$$

$$\phi_{*m}(z) = \frac{\mu r'^2\{\epsilon_1 - \epsilon_2 + i(\delta_1 + \delta_2)\}}{(K + 1)} \cdot \frac{1}{z}\,,$$

$$\psi_{*m}(z) = -\frac{2\mu r'^2(\epsilon_1 + \epsilon_2)}{(K + 1)\, z} + \frac{\mu r'^4\{\epsilon_1 - \epsilon_2 + i(\delta_1 + \delta_2)\}}{(K + 1)\, z^3}, \quad (46)$$

where the subscript i has been used for the inclusion and subscript m for the matrix.

The boundary condition satisfied on the outer boundary L_o is

$$\phi_o(t) + t\,\overline{\phi_o'(t)} + \overline{\psi_o(t)} = f_o(t)$$

where $f_o(t) = f(t) - \phi_{*m}(t) - t\,\overline{\phi'_{*m}(t)} - \overline{\psi_{*m}(t)}$; (47)

bar denotes the conjugate complex , and $f(t)$ accounts for the forces on the boundary. In the present case $f(t) = 0$.

So far, the origin was taken at the centre of L. Now a mapping function $z = \omega(\zeta) = \zeta - c$, $\zeta = \rho e^{i\vartheta}$ is applied which shifts the origin to O' , the centre of L_o .

The transformed equation of L_o is $|\zeta| = R$.
The boundary condition (47) is transformed to

$$\phi_{10}(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\phi_{10}'(\sigma)} + \overline{\psi_{10}(\sigma)} = f_{10}(\sigma) \tag{48}$$

where
$$\phi_{10}(\sigma) = \phi_o\{\omega(\sigma)\},$$
$$\psi_{10}(\sigma) = \psi_o\{\omega(\sigma)\},$$
$$f_{10}(\sigma) = f_o\{\omega(\sigma)\}$$

and $\zeta = Re^{i\vartheta} = R\sigma$ on L_o, $\phi_{10}(\zeta)$ and $\psi_{10}(\zeta)$ are functions analytic in the region $|\zeta| \leq R$.

These functions $\phi_{10}(\zeta)$ and $\psi_{10}(\zeta)$ may be found out using the methods discussed in chapter I.

The expressions of sectionally holomorphic functions $\phi_1(\zeta) = \phi\{\omega(\zeta)\}$ and $\psi_1(\zeta) = \psi\{\omega(\zeta)\}$ may be written using (37) and (38) . For matrix they are

$$\phi_1(\zeta) = -\frac{B(R\zeta + c_1\zeta^2)}{2R(R - c_1\zeta)} + \frac{(A_1 - iA_2)c_1\zeta^2}{(R - c_1\zeta)^2} - \frac{(A_1 - iA_2)\zeta^3}{R(R - c_1\zeta)^2} \tag{49}$$
$$+ \frac{(D_1 - iD_2)\zeta^3}{(R - c_1\zeta)^3} - \frac{(A_1 + iA_2)R}{(\zeta - c)} \qquad \text{for } |\zeta - c| > r'.$$

and

$$\psi_1(\zeta) = \frac{Bc_1(3R - c_1\zeta)}{2(R - c_1\zeta)} - \frac{2c_1R^2(A_1 - iA_2)}{(R - c_1\zeta)^2} - \frac{R^3(D_1 + iD_2)}{(\zeta - c)^3}$$

$$+ \frac{(A_1 - iA_2)(4R\zeta - 2c_1\zeta^2)}{(R - c_1\zeta)^2} - \frac{3(D_1 - iD_2)R^2\zeta}{(R - c_1\zeta)^3} + \frac{BR}{\zeta - c} \qquad (50)$$

for $|\zeta - c| > r'$,

where the following symbols have been used

$$B = - \frac{2\mu(\varepsilon_1 + \varepsilon_2)\, r'^2}{(K+1)\, R} ,$$

$$A_1 + iA_2 = - \frac{\mu\{\varepsilon_1 - \varepsilon_2 + i(\delta_1 + \delta_2)\}\, r'^2}{(K+1)\, R} , \qquad (51)$$

$$D_1 - iD_2 = - \frac{\mu\{\varepsilon_1 - \varepsilon_2 - i(\delta_1 + \delta_2)\}\, r'^4}{(K+1)\, R^3}$$

and $\quad c_1 = c/R$.

From (49) and (50) and the result given below in (52)

$$P_{\rho\rho} + P_{\vartheta\vartheta} = 4\,\mathrm{Re}[\,\phi_1'(\zeta)\,] ,$$

$$(52)$$

$$P_{\vartheta\vartheta} - P_{\rho\rho} + 2iP_{\rho\vartheta} = 2e^{2i\vartheta}\left[(\bar{\zeta} - c)\,\phi_1''(\zeta) + \psi_1'(\zeta)\right] ,$$

stresses for the matrix may be calculated.

It may be verified that the normal stress $P_{\rho\rho}$ and shearing

stress $P_{\rho\vartheta}$ vanish on the boundary of the circle $|\zeta| = R$, as they should.

The sectionally holomorphic functions $\phi(z)$ and $\psi(z)$ are

$$\phi_i(z) = -\frac{B\{R(z+c) + c_1(z+c)^2\}}{2R\{R - c_1(z+c)\}} + \frac{(A_1 - iA_2)c_1(z+c)^2}{\{R - c_1(z+c)\}^2}$$

$$-\frac{(A_1 - iA_2)(z+c)^3}{R\{R - c_1(z+c)\}^2} + \frac{(D_1 - iD_2)(z+c)^3}{\{R - c_1(z+c)\}^3}$$

$$-\frac{\mu\{\varepsilon_1 + \varepsilon_2 - i(\delta_1 - \delta_2)\}}{K+1} z,$$

$$\psi_i(z) = \frac{Bc_1\{3R - c_1(z+c)\}}{2\{R - c_1(z+c)\}} - \frac{2cR(A_1 - iA_2)}{\{R - c_1(z+c)\}^2} \tag{53}$$

$$+\frac{2(A_1 - iA_2)(z+c)\{2R - c_1(z+c)\}}{\{R - c_1(z+c)\}^2}$$

$$-\frac{3(D_1 - iD_2)R^2(z+c)}{\{R - c_1(z+c)\}^3} - \frac{(A_1 - iA_2)Rz}{r'^2},$$

$$\phi_m(z) = -\frac{B\{R(z+c) + c_1(z+c)^2\}}{2R\{R - c_1(z+c)\}} + \frac{(A_1 - iA_2)c_1(z+c)^2}{\{R - c_1(z+c)\}^2}$$

$$- \frac{(A_1 - iA_2)(z+c)^3}{R\{R - c_1(z+c)\}^2} + \frac{(D_1 - iD_2)(z+c)^3}{\{R - c_1(z+c)\}^3}$$

$$- \frac{(A_1 + iA_2)R}{z} ,$$

(54)

$$\psi_m(z) = \frac{Bc_1\{3R - c_1(z+c)\}}{2\{R - c_1(z+c)\}} - \frac{2cR(A_1 - iA_2)}{\{R - c_1(z+c)\}^2}$$

$$+ \frac{2(A_1 - iA_2)(z+c)\{2R - c_1(z+c)\}}{\{R - c_1(z+c)\}^2} - \frac{3(D_1 - iD_2)R^2(z+c)}{\{R - c_1(z+c)\}^3}$$

$$- \frac{(D_1 + iD_2)R^3}{z^3} + \frac{BR}{z} .$$

Stresses in the polar coordinates (r, θ) $(z = r\,e^{i\theta})$ are determined by using the formulas

$$P_{rr} + P_{\theta\theta} = 4\,\mathrm{Re}\,[\phi'(z)]$$

$$P_{\theta\theta} - P_{rr} + 2i\,P_{r\theta} = 2\,e^{2i\theta}\left[\bar{z}\,\phi''(z) + \psi'(z)\right] . \qquad (55)$$

Therefore , we get

$$(P_{rr}+P_{\theta\theta})_i = 4\ \mathrm{Re}\Bigg[-B\ \frac{\{R^2+2c(z+c)-c_1^2(z+c)^2\}}{2R\{R-c_1(z+c)\}^2}$$

$$+\frac{3(D_1-iD_2)R\ (z+c)^2}{\{R-c_1(z+c)\}^4}+\frac{2(A_1-iA_2)c(z+c)}{\{R-c_1(z+c)\}^3}$$

$$-\ \frac{(A_1-iA_2)(z+c)^2\{3R-c_1\ (z+c)\}}{R\{R-c_1(z+c)\}^3}$$

$$-\ \frac{\mu\{\epsilon_1+\epsilon_2-i(\delta_1-\delta_2)\}}{K+1}\Bigg], \tag{56}$$

and $(P_{rr}+P_{\theta\theta})_m = (P_{rr}+P_{\theta\theta})_i + 4\ \mathrm{Re}\Bigg[\dfrac{\mu\{\epsilon_1+\epsilon_2-i(\delta_1-\delta_2)\}}{K+1}$

$$+\frac{(A_1+iA_2)\ R}{z^2}\Bigg].$$

Also

$$(P_{\theta\theta}-P_{rr}+2i\ P_{r\theta})_i = 2\ e^{2i\theta}\Bigg[-\frac{2Bc\bar{z}}{\{R-c_1(z+c)\}^3}+\frac{B\ c^2}{R\{R-c_1(z+c)\}^2}$$

$$+\ \frac{6(D_1-iD_2)R\{R(z+c)+c_1(z+c)^2\}\ \bar{z}}{\{R-c_1(z+c)\}^5}$$

$$- \frac{3(D_1-iD_2)\,R^2\{R+2c_1(z+c)\}}{\{R-c_1(z+c)\}^4} + \frac{2c(A_1-iA_2)\bar{z}\,\{R+2c_1(z+c)\}}{\{R-c_1(z+c)\}^4}$$

$$- \frac{6(A_1-iA_2)R(z+c)}{\{R-c_1(z+c)\}^4} + \frac{4R^2(A_1-iA_2)}{\{R-c_1(z+c)\}^3} - \frac{4c^2(A_1-iA_2)}{\{R-c_1(z+c)\}^3}$$

$$- \left. \frac{(A_1 - iA_2)R}{r'^2} \right] \tag{57}$$

and $$(P_{\theta\theta}-P_{rr}+2i\,P_{r\theta})_m = (P_{\theta\theta}-P_{rr}+2i\,P_{r\theta})_i + 2e^{2i\theta}\left[\frac{(A_1-iA_2)R}{r'^2} - \frac{2(A_1+iA_2)R}{z^3} - \frac{BR}{z^2} + \frac{3(D_1+iD_2)R^3}{z^4}\right].$$

Continuity of normal stress P_{rr} and shearing stress $P_{r\theta}$ at the equilibrium interface may be verified using (56) and (57).

The displacement components in polar coordinates are given by the formula

$$2\mu(u_r+i\,u_\theta) = e^{-i\theta}\left[K\,\phi(z) - z\,\overline{\phi'(z)} - \overline{\psi(z)}\right].$$

So that

$$2\mu(u_r+iu_\theta)_i = e^{-i\theta}\Bigg[-\frac{B\,K}{2R}\,\frac{\{R(z+c)+c_1(z+c)^2\}}{\{R-c_1(z+c)\}}$$

$$+\frac{B\,z\,\{R^2+2c(\bar{z}+c)-c_1^2(\bar{z}+c)^2\}}{2R\{R-c_1(\bar{z}+c)\}^2}$$

$$-\frac{Bc_1\{3R-c_1(\bar{z}+c)\}}{2\{R-c_1(\bar{z}+c)\}}+\frac{K(A_1-iA_2)c_1(z+c)^2}{\{R-c_1(z+c)\}^2}$$

$$-\frac{K(A_1-iA_2)(z+c)^3}{R\{R-c_1(z+c)\}^2}-\frac{2c(A_1+iA_2)z(\bar{z}+c)}{\{R-c_1(\bar{z}+c)\}^3}$$

$$+\frac{(A_1+iA_2)z(\bar{z}+c)^2\{3R-c_1(\bar{z}+c)\}}{R\{R-c_1(\bar{z}+c)\}^3}+\frac{2(A_1+iA_2)cR}{\{R-c_1(\bar{z}+c)\}^2}$$

$$-\frac{2(A_1+iA_2)(\bar{z}+c)\{2R-c_1(\bar{z}+c)\}}{\{R-c_1(\bar{z}+c)\}^2}+\frac{K(D_1-iD_2)(z+c)^3}{\{R-c_1(z+c)\}^3}$$

$$-\frac{3(D_1+iD_2)R\,z(\bar{z}+c)^2}{\{R-c_1(\bar{z}+c)\}^4}+\frac{3(D_1+iD_2)R^2(\bar{z}+c)}{\{R-c_1(\bar{z}+c)\}^3}$$

$$-\frac{\mu K\{\epsilon_1+\epsilon_2-i(\delta_1-\delta_2)\}\,z}{K+1}$$

$$+\frac{\mu\{\epsilon_1+\epsilon_2+i(\delta_1-\delta_2)\}\,z}{K+1}+\frac{(A_1+iA_2)R\,\bar{z}}{r'^2}\Bigg]$$

$$\text{and} \quad 2\mu(u_r+iu_\theta)_m = 2\mu(u_r+iu_\theta)_i + \left[\frac{\mu K}{K+1}\{\epsilon_1+\epsilon_2-i(\delta_1-\delta_2)\} z - \frac{(A_1+iA_2)KR}{z} - \frac{\mu\{\epsilon_1+\epsilon_2+i(\delta_1-\delta_2)\} z}{K+1} - \frac{(A_1-iA_2)Rz}{\bar{z}2} - \frac{(A_1+iA_2)\, R\bar{z}}{r'^2} + \frac{(D_1-iD_2)R^3}{\bar{z}^3} - \frac{BR}{\bar{z}}\right] e^{-i\theta} \tag{58}$$

It may be noted that the relations (42) may be obtained from (58).

If the outer radius of the matrix R tends to infinity, the results for circular inclusion in an infinite medium are obtained((5)) . On putting $c = 0$, the results for concentric inclusion are obtained. The results of half plane may be obtained by taking $R - c = d$ where d is finite and R and c both tend to infinity ((6)).

In the above analysis the elastic constants for the inclusion and matrix were assumed to be the same. However, if the shear moduli for both the regions be the same but Poisson ratios be different, then the problem may be solved as follows :

The boundary conditions in this case shall be

$$\phi(t) + t\,\overline{\phi'(t)} + \overline{\psi(t)} = f(t) \quad \text{on } L_0 \tag{59}$$

$$\phi^+(t) + t\,\overline{\phi'^+(t)} + \overline{\psi^+(t)} = \phi^-(t) + t\,\overline{\phi'^-(t)} + \overline{\psi^-(t)} \quad \text{on } L \tag{60}$$

$$K\,\phi^{+}(t) - t\,\overline{\phi'^{+}(t)} - \overline{\psi^{+}(t)} = K_1\phi^{-}(t) - t\,\overline{\phi'^{-}(t)} - \overline{\psi^{-}(t)}$$

$$+\, 2\mu\, g(t) \text{ on } L \qquad (61)$$

where $K = 3 - 4\nu$ and $K_1 = 3 - 4\nu'$ for the plane strain case and $K = \frac{3-\nu}{1+\nu}$ and $K_1 = \frac{3-\nu'}{1+\nu'}$ for the generalized plane stress case, ν and ν' are the Poisson ratios of the inclusion and matrix respectively.

Adding (60) and (61), we get

$$\phi^{+}(t) - K'\,\phi^{-}(t) = \frac{2\mu\, g(t)}{K+1} \text{ on } L \qquad (62)$$

where $K' = \frac{K_1+1}{K+1}$.

From (60)

$$\psi^{+}(t) - \psi^{-}(t) = -\left[\overline{\phi^{+}(t)} - \overline{\phi^{-}(t)}\right] - \bar{t}\left[\phi'^{+}(t) - \phi'^{-}(t)\right]$$

$$\text{on } L. \qquad (63)$$

Equation (62) may be reduced to the equation of the form

$$F^{+}(t) - F^{-}(t) = \frac{2\,\mu\, g(t)}{K+1} \qquad (64)$$

when $\phi(z)$ is defined as

$$\phi^{+}(z) = F^{+}(z) \text{ and } K'\,\phi^{-}(z) = F^{-}(z)\,. \qquad (65)$$

Having found $\phi^+(z)$ and $\phi^-(z)$, $\psi^+(z)$ and $\psi^-(z)$ may be determined from (62) and the problem is solved.

Stresses were calculated numerically and graphs of normal stress, shearing stress and hoop stress both for inclusion and matrix at the equilibrium boundary have been given in the appendix following this chapter. In each case θ varies from 0 to π. Two cases $\epsilon_1=\epsilon_2=\epsilon$ and $\epsilon_1=-\epsilon_2=\epsilon$ have been considered. As a typical case the radii of inclusion and matrix are in the ratio 1 : 4. Let the ratio of the distance c of the centre of the matrix to the radius of the inclusion be denoted by C. Graphs have been drawn for six values of C : 0.0, 0.5, 1.0, 1.5, 2.0 and 2.5. Comparison of concentric and eccentric case has been done whenever the difference was significant.

It is interesting to note that when $\epsilon_1=\epsilon_2=\epsilon$ and the inclusion is very near the outer boundary, the hoop stress in inclusion at the equilibrium boundary becomes positive for some values of θ and is nearly zero when C = 2.7 . For $\epsilon_1=-\epsilon_2=\epsilon$, normal stress shows an interesting behaviour when the inclusion is sufficiently near the outer boundary.

APPENDIX TO CHAPTER II

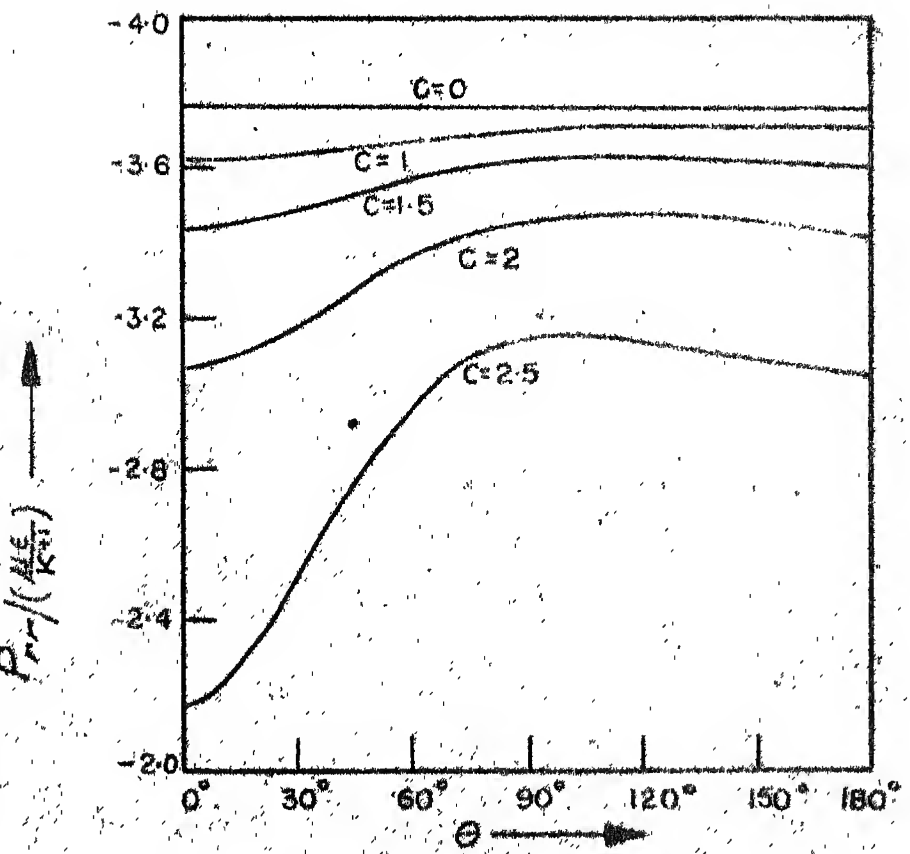


Fig.1. Normal stress at the equilibrium boundary for $\epsilon_1=\epsilon_2=\epsilon$.

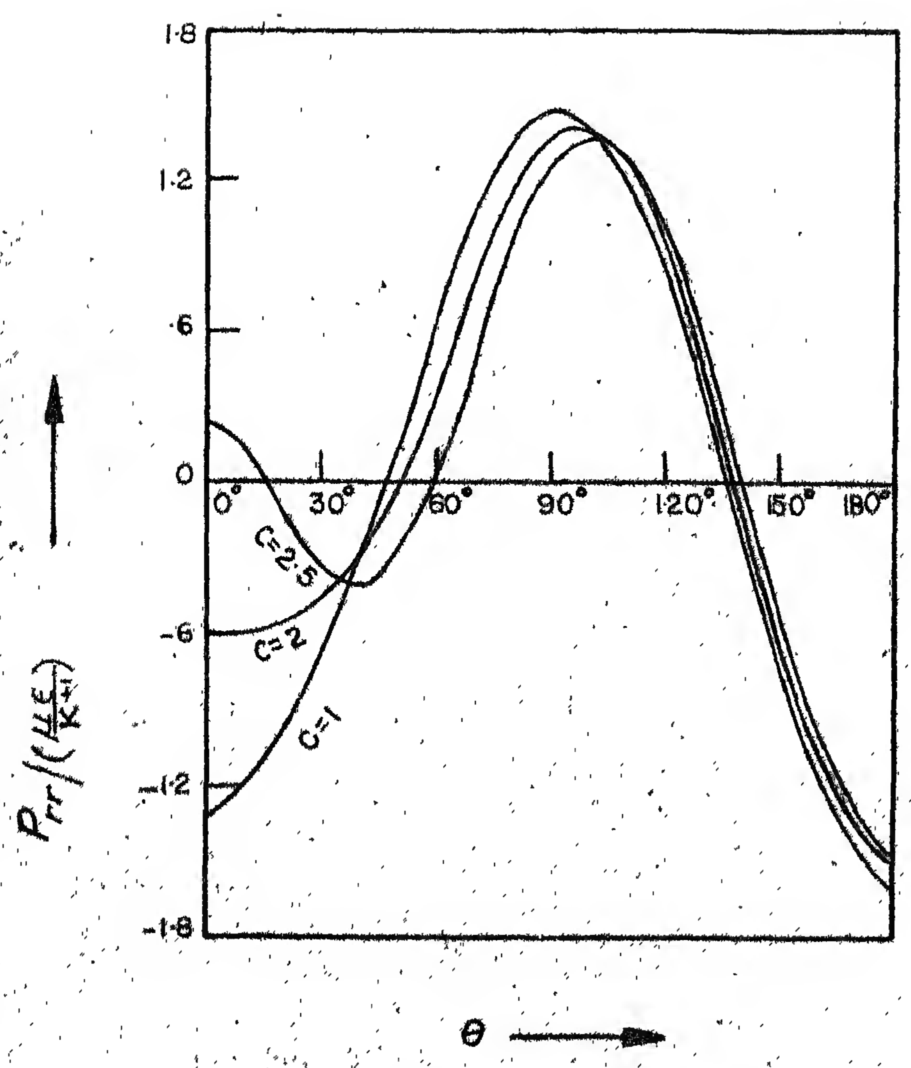


Fig.2 . Normal stress at the equilibrium boundary for $\epsilon_1 = -\epsilon_2 = \epsilon$

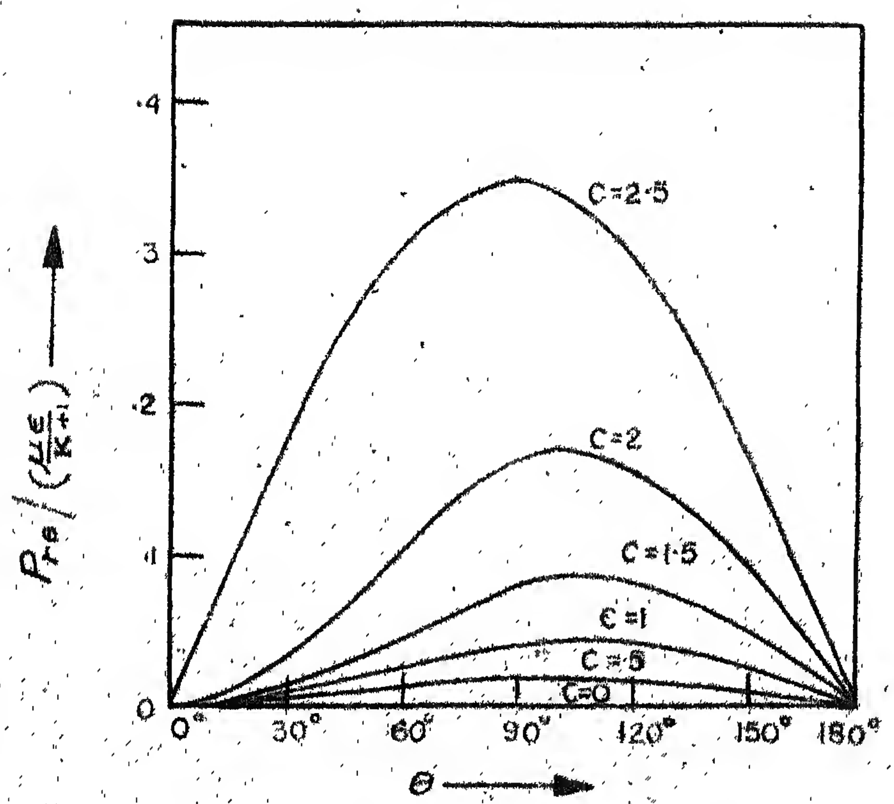


Fig. 3. Tangential stress at the equilibrium boudary for $\epsilon_1 = \epsilon_2 = \epsilon$

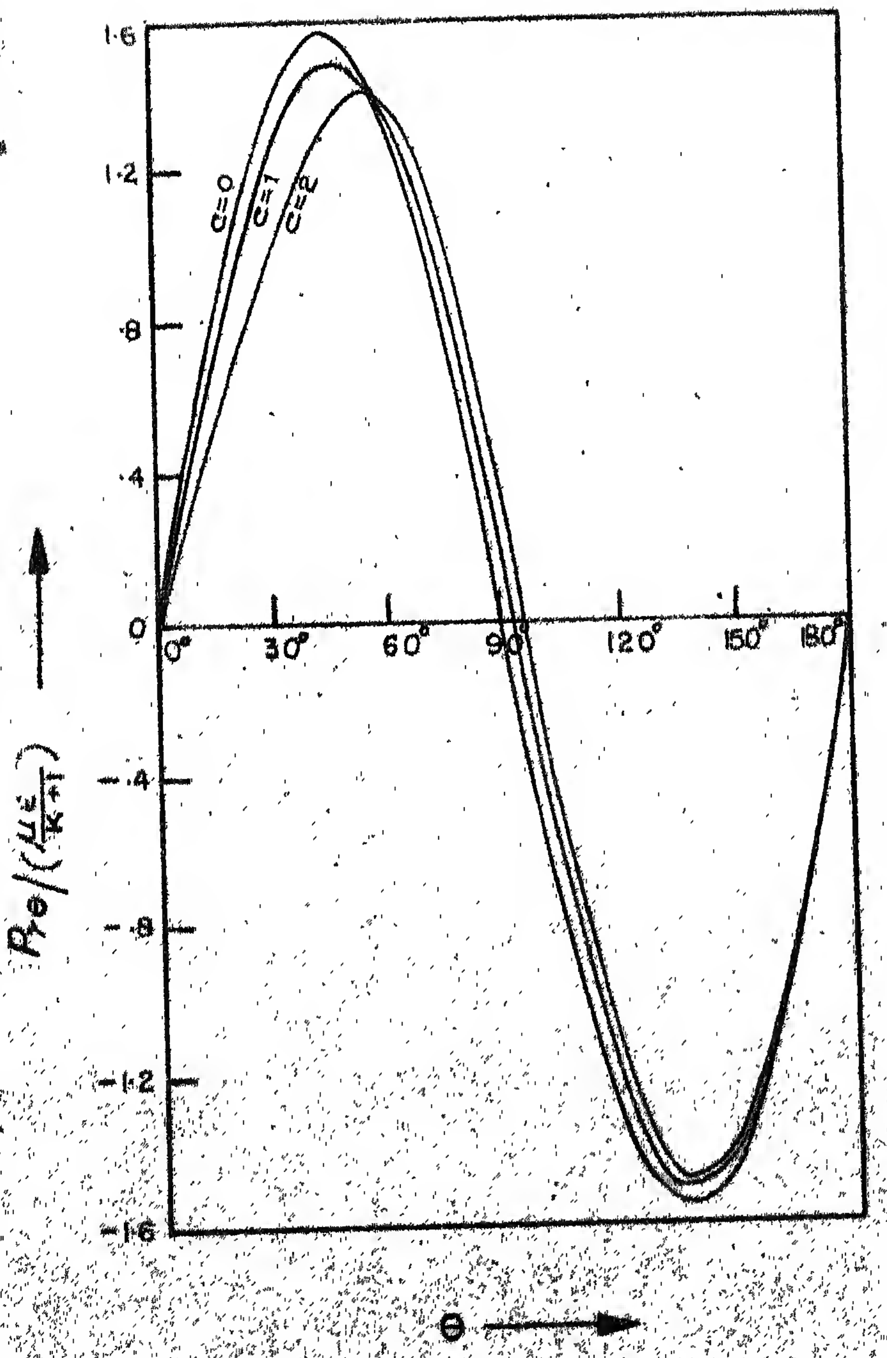


Fig.4. Tangential stress at the equilibrium boundary for $\epsilon_1 = -\epsilon_2 = \epsilon$

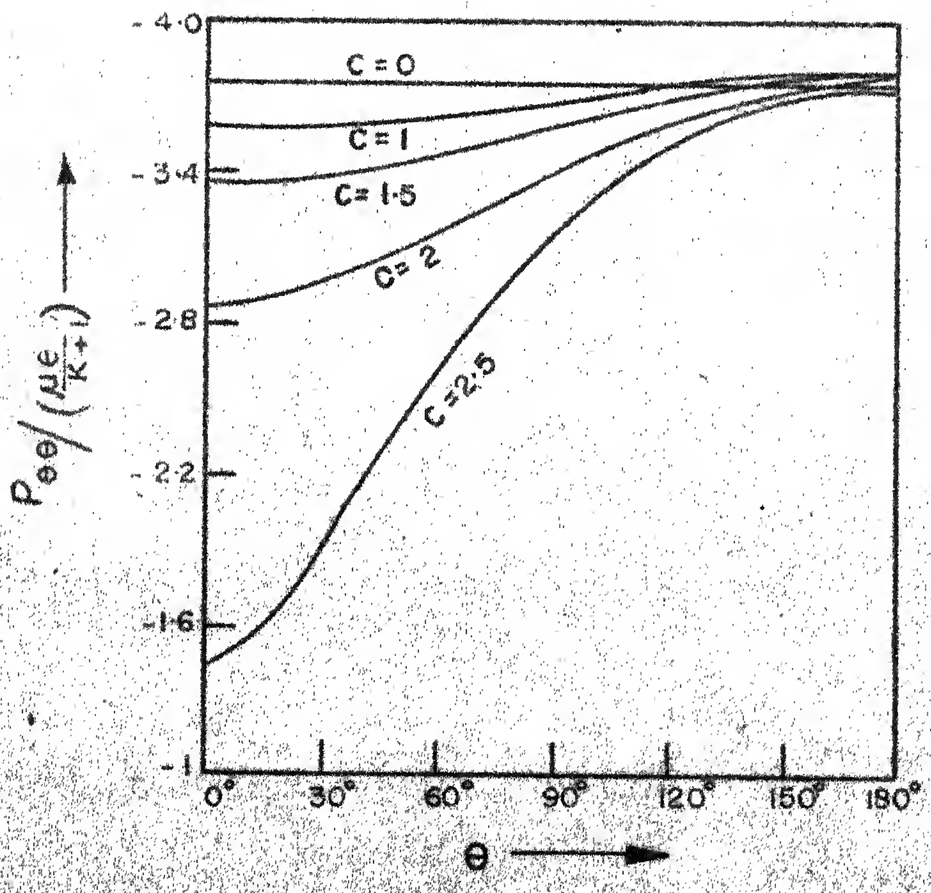


Fig.5. Hoop stress inside at the equilibrium boundray for $\epsilon_1 = \epsilon_2 = \epsilon$

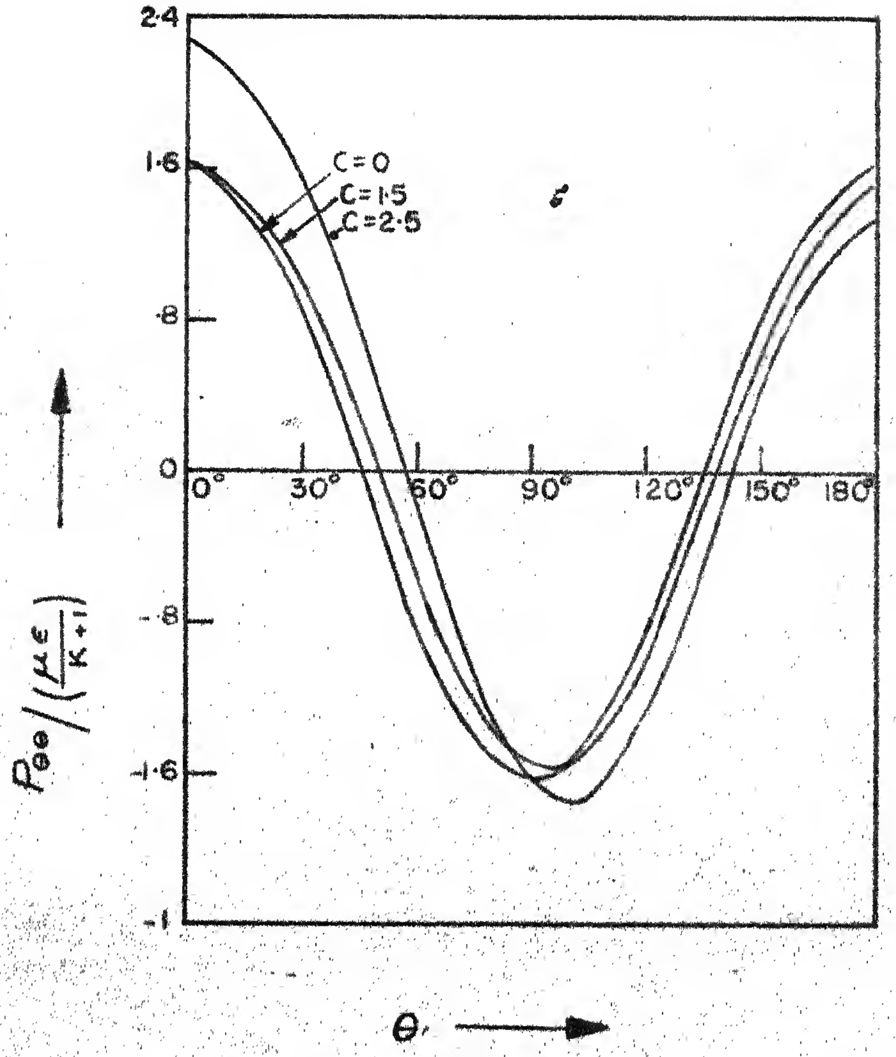


Fig.6. Hoop stress inside at the equilibrium boundary for $\epsilon_1 = -\epsilon_2 = \epsilon$

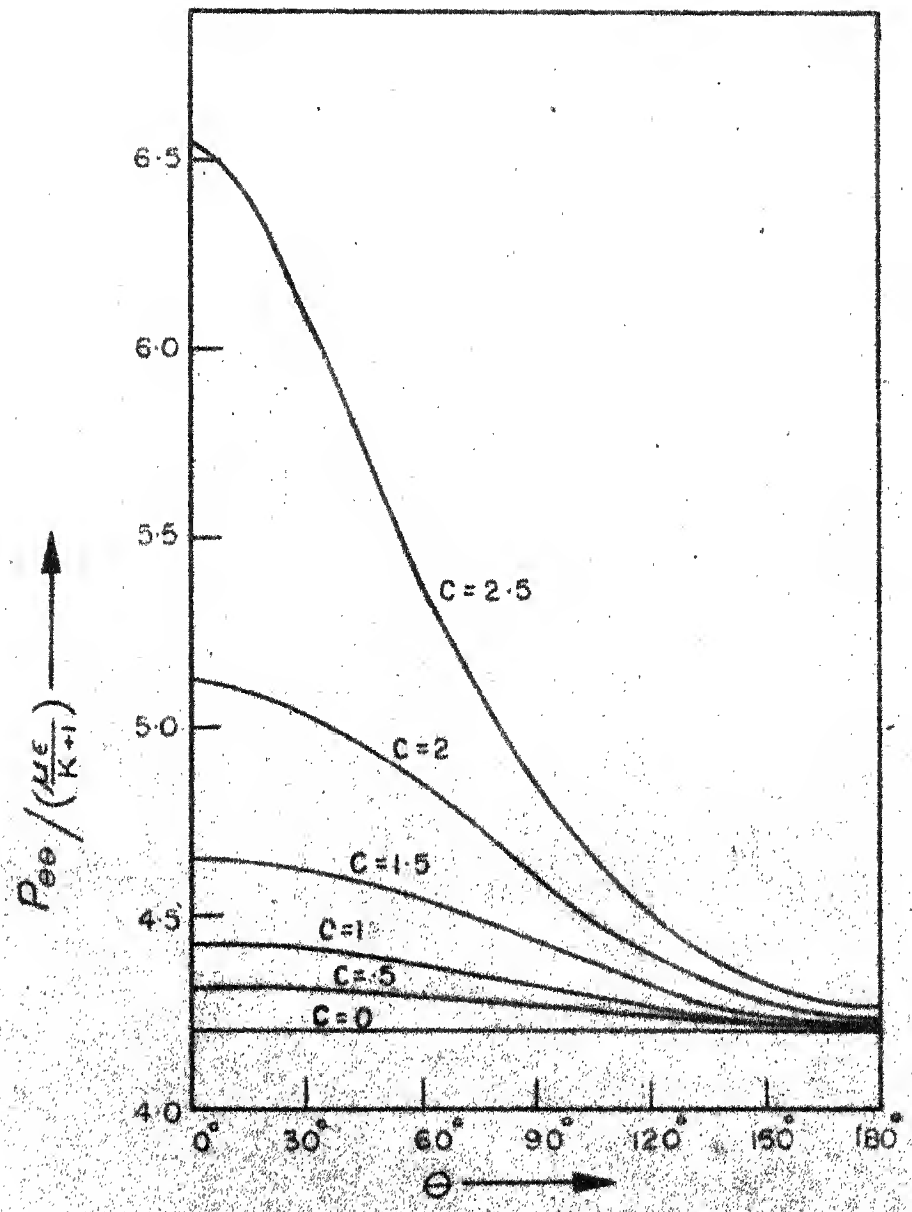


Fig. 7. Hoop stress outside at the eqilibrium boundray for $\epsilon_1 = \epsilon_2 = \epsilon$.

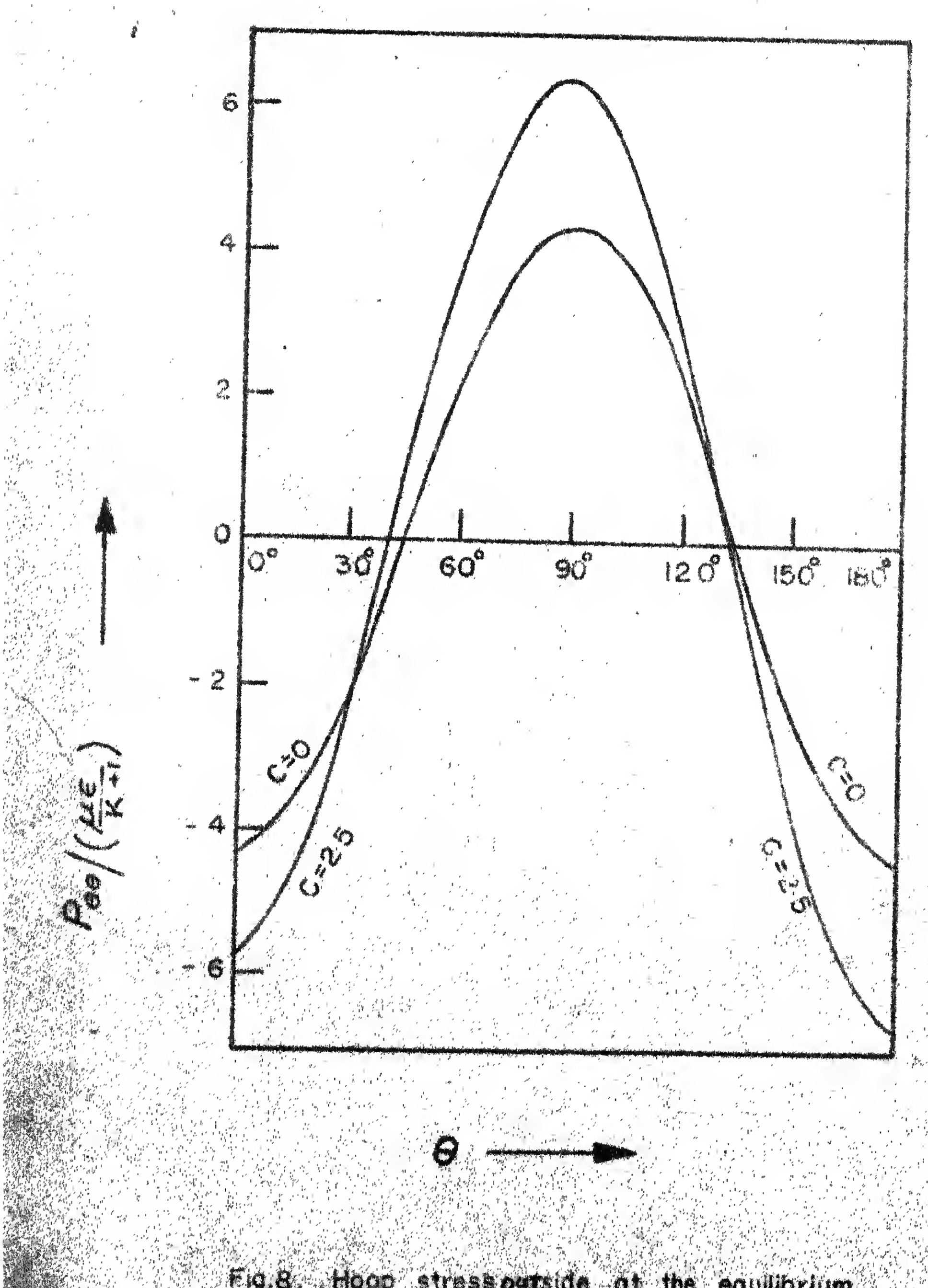


Fig.8. Hoop stress outside at the equilibrium boundray for $\epsilon_1 = -\epsilon_2 = \epsilon$

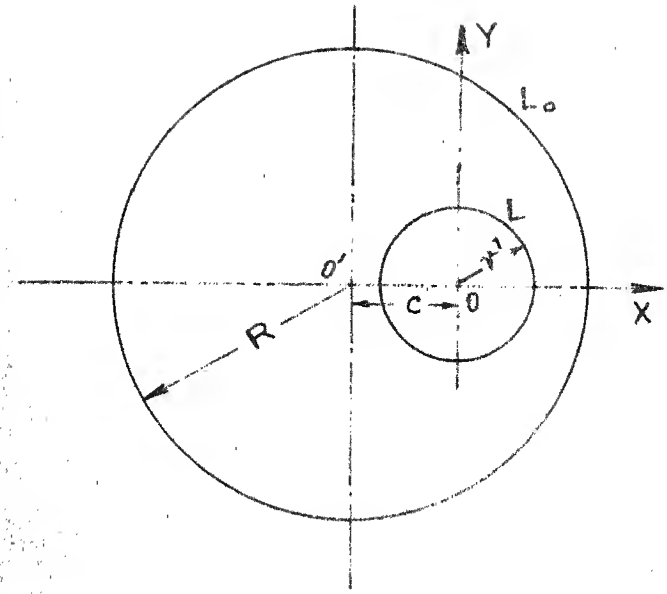


FIG.-1- ECCENTRIC CIRCULAR INCLUSION IN A CIRCULAR REGION.

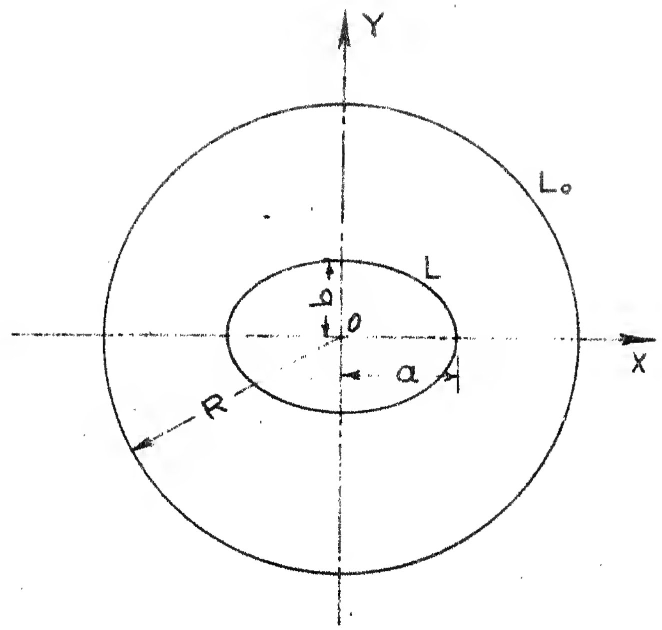


FIG.-2- ELLIPTICAL INCLUSION IN A CIRCULAR REGION.

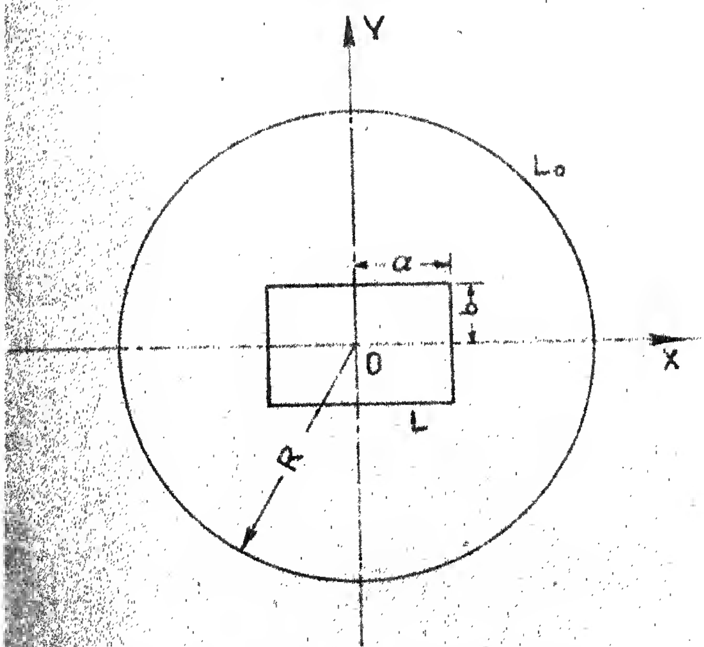


FIG.-3-.RECTANGULAR INCLUSION IN A CIRCULAR REGION.

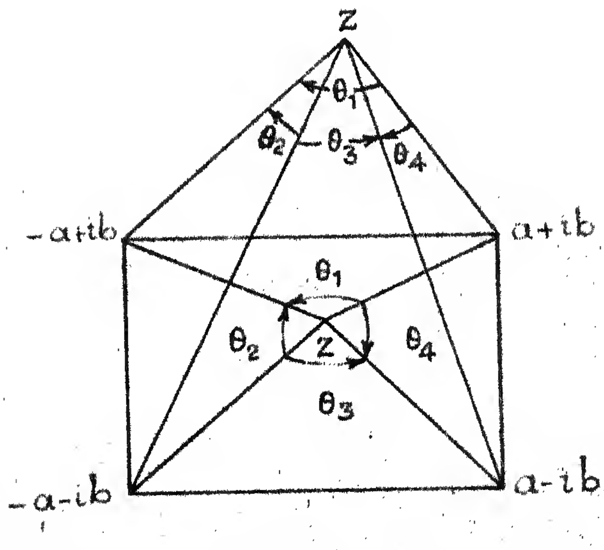


FIG.-4- THE ANGLES $\theta_1, \theta_2, \theta_3$ AND θ_4 FOR INCLUSION AND MATRIX.

CHAPTER III

ELLIPTIC INCLUSION IN A CIRCULAR REGION

This chapter deals with the problem of misfitting concentric elliptic inclusion in a circular region. The problem investigated may be stated as follows :

Let a circular ring of outer radius R have a concentric elliptic hole. The semi-major and semi-minor axes of this elliptic hole are a and b respectively (Figure 2 p.48). This medium is called matrix. Now in this matrix, an elastic solid of dimensions slightly larger than those of the hole but remaining within the limits of proportional elasticity are embedded. The elastic field developed due to misfit in size is evaluated.

We use the terminology of chapter II. As stated there the outer boundary of the matrix is denoted by L_o

and the inner boundary is denoted by L. The centre of L_o is taken as the origin and is denoted by O. In the absence of the surrounding material, the inclusion misfits by a displacement $(\epsilon_1 x + \delta_1 y \, , \, \epsilon_2 y + \delta_2 x)$ with respect to the origin at O.

If the displacement components in Cartesian coordinates at the equilibrium boundary for inclusion and matrix are denoted by $(u_b^+ \, , \, v_b^+)$ and $(u_b^- \, , \, v_b^-)$ respectively, then

$$
\begin{aligned}
u_b^+ - u_b^- &= - \; (\epsilon_1 x + \delta_1 y) = g_1(t) \, , \\
v_b^+ - v_b^- &= - \; (\epsilon_2 y + \delta_2 x) = g_2(t) \, .
\end{aligned}
\tag{66}
$$

It may be noted that $(u_b^+ \, , \, v_b^+)$, the displacement at the boundary of inclusion is measured from its natural state ; the state it would have in the absence of the matrix ; while $(u_b^- \, , \, v_b^-)$ is the displacement at the boundary of matrix measured from its natural state, which is the state when its inner boundary is an ellipse of semi-major and semi-minor axes a and b respectively.

The equation of L_o is $|z| = R$. For a point on L, z is denoted by t. Substituting (66) in (33) and (36), it may be seen that

$$g(t) = -\{\epsilon_1+\epsilon_2-i(\delta_1-\delta_2)\}\frac{t}{2} - \{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\}\frac{\bar{t}}{2},$$

(67)

$$h(t) = (\epsilon_1+\epsilon_2)\,\bar{t} + \{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\}\frac{t}{2} + $$
$$+ \{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\}\frac{\bar{t}}{2}\frac{\overline{dt}}{dt}.$$

Now putting the values of $g(t)$ and $h(t)$ from (67) in (39) and (40) and evaluating the contour integrals, the expressions of $\phi_*(z)$ and $\psi_*(z)$ for inclusion and matrix come out to be

$$\phi_{*i}(z) = -\frac{\mu}{(K+1)}\{\epsilon_1+\epsilon_2-i(\delta_1-\delta_2)\}z - \frac{\mu}{(K+1)}\frac{(a-b)}{(a+b)}\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\}z,$$

(68)

$$\psi_{*i}(z) = \frac{\mu}{(K+1)}\left[\frac{2(\epsilon_1+\epsilon_2)(a-b)}{(a+b)} + \{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\}\frac{(a-b)^2}{(a+b)^2}\right.$$
$$\left. + \{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\}\right]z,$$

$$\phi_{*m}(z) = \frac{2\mu}{(K+1)}\frac{a\,b}{(a^2-b^2)}\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\}\{z-(z^2-c^2)^{1/2}\},$$

(69)

$$\psi_{*m}(z) = -\frac{4\mu}{(K+1)}\frac{a\,b}{(a^2-b^2)}(\epsilon_1+\epsilon_2)\{z-(z^2-c^2)^{1/2}\}$$
$$-\frac{2\mu}{(K+1)}\frac{ab(a^2+b^2)}{(a^2-b^2)^2}\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\}\{2(z-(z^2-c^2)^{1/2})$$
$$-\frac{c^2}{(z^2-c^2)^{1/2}}\},$$

where $c^2 = a^2 - b^2$.

The boundary condition satisfied on the outer boundary L_o is given by (41). In the present case $f(t)=0$. Using (69), $f_o(t)$ can be found out. The value of $f_o(t)$ is substituted in (41) and then the functions $\phi_o(z)$ and $\psi_o(z)$ which are analytic in the region $|z| \leqslant R$ may be determined using the methods given in chapter I. The analytic functions $\phi_o(z)$ and $\psi_o(z)$ are

$$\phi_o(z) = \frac{\mu}{(K+1)} \frac{2ab}{(a^2-b^2)}\Bigg[\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\}\left\{\frac{R^2}{(R^4-c^2z^2)^{1/2}} - 1\right\}z$$

$$- 2(\epsilon_1+\epsilon_2)\left\{(R^4-c^2z^2)^{1/2} - R^2 + \frac{c^2z^2}{4R^2}\right\}\cdot\frac{1}{z}$$

$$- \frac{2(a^2+b^2)}{(a^2-b^2)}\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\}\left\{(R^4-c^2z^2)^{1/2} - R^2 + \frac{c^2z^2}{4R^2}\right\}\cdot\frac{1}{z}$$

$$- (a^2+b^2)\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\}\left\{\frac{z}{(R^4-c^2z^2)^{1/2}} - \frac{z}{2R^2}\right\}\Bigg] \tag{70}$$

and

$$\psi_o(z) = \frac{\mu}{(K+1)} \frac{2ab}{(a^2-b^2)}\Bigg[\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\}\Big\{(R^4-c^2z^2)^{1/2} -$$

$$- \frac{R^8}{(R^4-c^2z^2)^{3/2}}\Big\}\cdot\frac{1}{z} - 2(\epsilon_1+\epsilon_2)\Big\{\frac{R^6}{(R^4-c^2z^2)^{1/2}}$$

$$-R^4 - \frac{c^2z^2}{2}\Big\}\cdot\frac{1}{z^3} -$$

$$-\frac{2(a^2+b^2)}{(a^2-b^2)}\left\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\right\}\left\{\frac{R^6}{(R^4-c^2z^2)^{1/2}}-R^4-\right.$$

$$\left.-\frac{c^2\,z^2}{2}\right\}\frac{1}{z^3}+(a^2+b^2)\left\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\right\}.$$

$$\left.\left\{\frac{R^6}{(R^4-c^2z^2)^{3/2}}-1\right\}\cdot\frac{1}{z}\right].$$

The sectionally holomorphic functions $\phi(z)$ and $\psi(z)$ may be found out using (37) and (38) and are given below.

$$\phi_i(z)=\phi_o(z)-\frac{\mu}{K+1}\left[\left\{\epsilon_1+\epsilon_2-i(\delta_1-\delta_2)\right\}+\frac{(a-b)}{(a+b)}\left\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\right\}\right]z,$$

$$\psi_i(z)=\psi_o(z)+\frac{\mu}{K+1}\left[2(\epsilon_1+\epsilon_2)\frac{(a-b)}{(a+b)}+\left\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\right\}\right. \tag{71}$$

$$\left.+\frac{(a-b)^2}{(a+b)^2}\left\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\right\}\right]z,$$

$$\phi_m(z)=\phi_o(z)+\frac{2\mu}{(K+1)}\frac{ab}{(a^2-b^2)}\left\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\right\}\left\{z-(z^2-c^2)^{1/2}\right\},$$

$$\tag{72}$$

$$\psi_m(z)=\psi_o(z)-\frac{2ab}{(a^2-b^2)}\frac{\mu}{(K+1)}\left[2(\epsilon_1+\epsilon_2)\left\{z-(z^2-c^2)^{1/2}\right\}\right.$$

$$\left.+\frac{(a^2+b^2)}{(a^2-b^2)}\left\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\right\}\left\{2\left(z-(z^2-c^2)^{1/2}\right)-\frac{c^2}{(z^2-c^2)^{1/2}}\right\}\right].$$

Stresses in Cartesian coordinate system are determined using the formulas given in (9). They are

$$(P_{xx}+P_{yy})_i = \frac{\mu}{K+1}\frac{4\ ab}{(a^2-b^2)}\Big[\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\}\Big\{\frac{R^6}{(R^4-c^2z^2)^{3/2}}-1\Big\}$$

$$-\ 2(\epsilon_1+\epsilon_2)R^2\Big\{-\frac{R^2}{z^2(R^4-c^2z^2)^{1/2}}+\frac{1}{z^2}+\frac{c^2}{4R^4}\Big\}$$

$$-\frac{2(a^2+b^2)}{(a^2-b^2)}R^2\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\}\cdot$$

$$\Big\{-\frac{R^2}{z^2(R^4-c^2z^2)^{1/2}}+\frac{1}{z^2}+\frac{c^2}{4R^4}\Big\}$$

$$-(a^2+b^2)\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\}\Big\{\frac{R^4}{(R^4-c^2z^2)^{3/2}}-\frac{1}{2R^2}\Big\}$$

$$+\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\}\Big\{\frac{R^6}{(R^4-c^2\bar{z}^2)^{3/2}}-1\Big\}$$

$$-\ 2(\epsilon_1+\epsilon_2)R^2\Big\{-\frac{R^2}{\bar{z}^2(R^4-c^2\bar{z}^2)^{1/2}}+\frac{1}{\bar{z}^2}+\frac{c^2}{4R^4}\Big\}$$

$$-\frac{2(a^2+b^2)}{(a^2-b^2)}R^2\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\}\cdot$$

$$\Big\{-\frac{R^2}{\bar{z}^2(R^4-c^2\bar{z}^2)^{1/2}}+\frac{1}{\bar{z}^2}+\frac{c^2}{4R^4}\Big\}$$

$$-(a^2+b^2)\left\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\right\}\left\{\frac{R^4}{(R^4-c^2\bar{z}^2)^{3/2}}-\frac{1}{2R^2}\right\}\Bigg]$$

$$-\frac{4\mu}{K+1}\left\{\epsilon_1+\epsilon_2+\frac{(a-b)}{(a+b)}(\epsilon_1-\epsilon_2)\right\},$$

(73)

$$(P_{xx}+P_{yy})_m=(P_{xx}+P_{yy})_i+\frac{4\mu}{K+1}\left\{\epsilon_1+\epsilon_2+\frac{(a-b)}{(a+b)}(\epsilon_1-\epsilon_2)\right\}$$

$$+\frac{\mu}{K+1}\frac{4ab}{(a^2-b^2)}\left\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\right\}\left\{1-\frac{z}{(z^2-c^2)^{1/2}}\right\}$$

$$+\frac{\mu}{K+1}\frac{4ab}{(a^2-b^2)}\left\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\right\}\left\{1-\frac{\bar{z}}{(\bar{z}^2-c^2)^{1/2}}\right\}$$

and

$$(P_{yy}-P_{xx}+2iP_{xy})_i=\frac{\mu}{K+1}\frac{4ab}{(a^2-b^2)}\Bigg[\left\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\right\}\cdot$$

$$\frac{\left\{3R^6c^2z\,\bar{z}-R^4(c^4z^2+2R^4c^2)\right\}}{(R^4-c^2z^2)^{5/2}}-2(\epsilon_1+\epsilon_2)\cdot$$

$$\left\{\frac{R^4(2R^4-3c^2z^2)\bar{z}}{z^3(R^4-c^2z^2)^{3/2}}-\frac{2R^2\bar{z}}{z^3}\right.$$

$$\left.+\frac{R^6(4c^2z^2-3R^4)}{z^4(R^4-c^2z^2)^{3/2}}+\frac{c^2}{2z^2}+\frac{3R^4}{z^4}\right\}-\frac{2(a^2+b^2)}{(a^2-b^2)}\cdot$$

$$\left\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\right\}\left\{\frac{R^4(2R^4-3c^2z^2)\bar{z}}{z^3(R^4-c^2z^2)^{3/2}}-\frac{2R^2\bar{z}}{z^3}\right.$$

$$\left.+\frac{R^6(4c^2z^2-3R^4)}{z^4(R^4-c^2z^2)^{3/2}}+\frac{c^2}{2z^2}+\frac{3R^4}{z^4}\right\}-$$

$$-(a^2+b^2)\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\}\left\{\frac{3R^4c^2z\,\bar{z}}{(R^4-c^2z^2)^{5/2}} - \frac{R^6(4c^2z^2-R^4)}{z^2(R^4-c^2z^2)^{5/2}} - \frac{1}{z^2}\right\}\Bigg]$$

$$+ \frac{2\mu}{K+1}\left[\frac{2(a-b)}{(a+b)}(\epsilon_1+\epsilon_2)+\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\}\frac{(a-b)^2}{(a+b)^2} +\right.$$

$$\left.+\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\}\right],$$

$$(74)$$

$$(P_{yy}-P_{xx}+2iP_{xy})_m = (P_{yy}-P_{xx}+2iP_{xy})_i - \frac{2\mu}{K+1}\left[\frac{2(a-b)}{(a+b)}(\epsilon_1+\epsilon_2)\right.$$

$$\left.+\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\}\frac{(a-b)^2}{(a+b)^2} + \{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\}\right]$$

$$+\frac{\mu}{(K+1)}\frac{4ab}{(a^2-b^2)}\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\}\left[\frac{c^2}{(z^2-c^2)^{3/2}}\right.$$

$$\left.-\frac{(a^2+b^2)}{(a^2-b^2)}\left\{2\left(1-\frac{z}{(z^2-c^2)^{1/2}}\right)+\frac{c^2z}{(z^2-c^2)^{3/2}}\right\}\right]$$

$$-\frac{\mu}{(K+1)}\frac{8ab}{(a^2-b^2)}(\epsilon_1+\epsilon_2)\left\{1-\frac{z}{(z^2-c^2)^{1/2}}\right\}.$$

The displacement components in Cartesian coordinates are given by (10). For inclusion and the matrix, they are

$$2\mu(u+iv)_i = \frac{\mu}{K+1}\ \frac{2ab}{(a^2-b^2)}\Big[K\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\}\Big\{\frac{R^2}{(R^4-c^2z^2)^{1/2}}-1\Big\}z$$

$$-\ 2K(\epsilon_1+\epsilon_2)\Big\{(R^4-c^2z^2)^{1/2}-R^2+\frac{c^2z^2}{4R^2}\Big\}\cdot\ \frac{1}{z}$$

$$-2\ \frac{(a^2+b^2)}{(a^2-b^2)}\ K\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\}\Big\{(R^4-c^2z^2)^{1/2}-R^2$$

$$+\ \frac{c^2z^2}{4R^2}\Big\}\cdot\ \frac{1}{z}$$

$$-K(a^2+b^2)\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\}\Big\{\frac{1}{(R^4-c^2z^2)^{1/2}}-\frac{1}{2R^2}\Big\}z$$

$$-\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\}\Big\{\frac{R^6}{(R^4-c^2\bar{z}^2)^{3/2}}-1\Big\}z$$

$$+\ 2(\epsilon_1+\epsilon_2)\Big\{-\frac{R^4}{\bar{z}^2(R^4-c^2\bar{z}^2)^{1/2}}+\frac{R^2}{\bar{z}^2}+\frac{c^2}{4R^2}\Big\}z$$

$$+\ \frac{2(a^2+b^2)}{(a^2-b^2)}\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\}\Big\{-\frac{R^4}{\bar{z}^2(R^4-c^2\bar{z}^2)^{1/2}}$$

$$+\ \frac{R^2}{\bar{z}^2}+\frac{c^2}{4R^2}\Big\}z\quad +(a^2+b^2)\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\}\cdot$$

$$\Big\{\frac{R^4}{(R^4-c^2\bar{z}^2)^{3/2}}-\frac{1}{2R^2}\Big\}z-\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\}\cdot$$

$$\Big\{(R^4-c^2\bar{z}^2)^{1/2}-\frac{R^8}{(R^4-c^2\bar{z}^2)^{3/2}}\Big\}\cdot\ \frac{1}{\bar{z}}+$$

$$+ \quad 2(\epsilon_1 + \epsilon_2)\left\{\frac{R^6}{(R^4-c^2\bar{z}^2)^{1/2}} - R^4 - \frac{c^2\,\bar{z}^2}{2}\right\}. \quad \frac{1}{\bar{z}^3}$$

$$+ \quad \frac{2(a^2+b^2)}{(a^2-b^2)}\left\{\epsilon_1 - \epsilon_2 + i(\delta_1+\delta_2)\right\}\left\{\frac{R^6}{(R^4-c^2\bar{z}^2)^{1/2}} - R^4 - \frac{c^2\bar{z}^2}{2}\right\}. \frac{1}{\bar{z}^3}$$

$$-(a^2+b^2)\left\{\epsilon_1 - \epsilon_2 + i(\delta_1+\delta_2)\right\}\left\{\frac{R^6}{(R^4-c^2\bar{z}^2)^{3/2}} - 1\right\}. \quad \frac{1}{\bar{z}} \Bigg]$$

$$- \frac{\mu K}{K+1}\left[\left\{\epsilon_1 + \epsilon_2 - i(\delta_1-\delta_2)\right\} + \frac{(a-b)}{(a+b)}\left\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\right\}\right] z$$

$$+ \quad \frac{\mu}{K+1} \quad \left[\left\{\epsilon_1+\epsilon_2+i(\delta_1-\delta_2)\right\} \quad + \quad \frac{(a-b)}{(a+b)} \quad \left\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\right\}\right] \, z$$

$$- \frac{2\mu}{K+1} \quad \frac{(a-b)}{(a+b)} \, (\epsilon_1+\epsilon_2) \; \bar{z} \quad - \frac{\mu}{K+1} \, \frac{(a-b)^2}{(a+b)^2} \left\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\right\} \bar{z}$$

$$- \frac{\mu}{K+1} \left\{\epsilon_1 - \epsilon_2 + i(\delta_1 + \delta_2)\right\} \; \bar{z} \tag{75}$$

and

$$2\mu(u+iv)_m = 2\mu(u+iv)_i + \frac{\mu K}{K+1} \left[\left\{\epsilon_1+\epsilon_2-i(\delta_1-\delta_2)\right\} + \frac{(a-b)}{(a+b)} \,.\right.$$

$$\left.\left\{\epsilon_1 - \epsilon_2 + i(\delta_1 + \delta_2)\right\}\right] z \quad -$$

$$- \frac{\mu}{K+1}\left[\{\epsilon_1+\epsilon_2+i(\delta_1-\delta_2)\} + \frac{(a-b)}{(a+b)}\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\}\right] z$$

$$+ \frac{2\mu}{K+1}\frac{(a-b)}{(a+b)}(\epsilon_1+\epsilon_2)\bar{z} + \frac{\mu}{K+1}\frac{(a-b)^2}{(a+b)^2}\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\}\bar{z}$$

$$+ \frac{\mu}{K+1}\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\}\bar{z} + \frac{\mu K}{K+1}\frac{2ab}{(a^2-b^2)}\{\epsilon_1-\epsilon_2+i(\delta_1+\delta_2)\} \cdot$$

$$\{z-(z^2-c^2)^{1/2}\} - \frac{\mu}{K+1}\frac{2ab}{(a^2-b^2)}\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\}\left\{1-\frac{\bar{z}}{(\bar{z}^2-c^2)^{1/2}}\right\}$$

$$+ \frac{\mu}{K+1}\frac{4ab}{a^2-b^2}(\epsilon_1+\epsilon_2)\{\bar{z}-\sqrt{\bar{z}^2-c^2}\} + \frac{\mu}{K+1}\frac{2ab(a^2+b^2)}{(a^2-b^2)^2} \cdot$$

$$\{\epsilon_1-\epsilon_2-i(\delta_1+\delta_2)\}\left[2\{\bar{z}-(\bar{z}^2-c^2)^{1/2}\} - \frac{c^2}{(\bar{z}^2-c^2)^{1/2}}\right] .$$

It may be noted that (66) may be obtained from (75).

The normal and shearing stresses on the outer boundary L_o vanish which may be verified using (73) and (74). In order to verify the continuity of normal and shearing stress across the equilibrium interface i.e. L, the transformation

$$z = c\cosh\zeta \quad , \quad \zeta = \xi + i\eta \qquad (76)$$

is used ((4)) . The boundary of ellipse $x^2/a^2 + y^2/b^2 = 1$ is now defined by $\xi = \xi_0$ where ξ_0 and c are given by the equations

$$c \cosh \xi_0 = a$$

$$\text{and} \quad c \sinh \xi_0 = b \quad . \tag{77}$$

If the normal, shearing and hoop stresses are denoted by $P_{\xi\xi}$, $P_{\xi\eta}$ and $P_{\eta\eta}$ respectively, then

$$P_{\xi\xi} + P_{\eta\eta} = 4\,\mathrm{Re}\left[\,\phi'(z)\right]$$

$$\text{and} \qquad P_{\eta\eta} - P_{\xi\xi} + 2iP_{\xi\eta} = \frac{2\sinh \zeta}{\sinh \bar{\zeta}}\left[\bar{z}\,\phi''(z) + \psi'(z)\right] \tag{78}$$

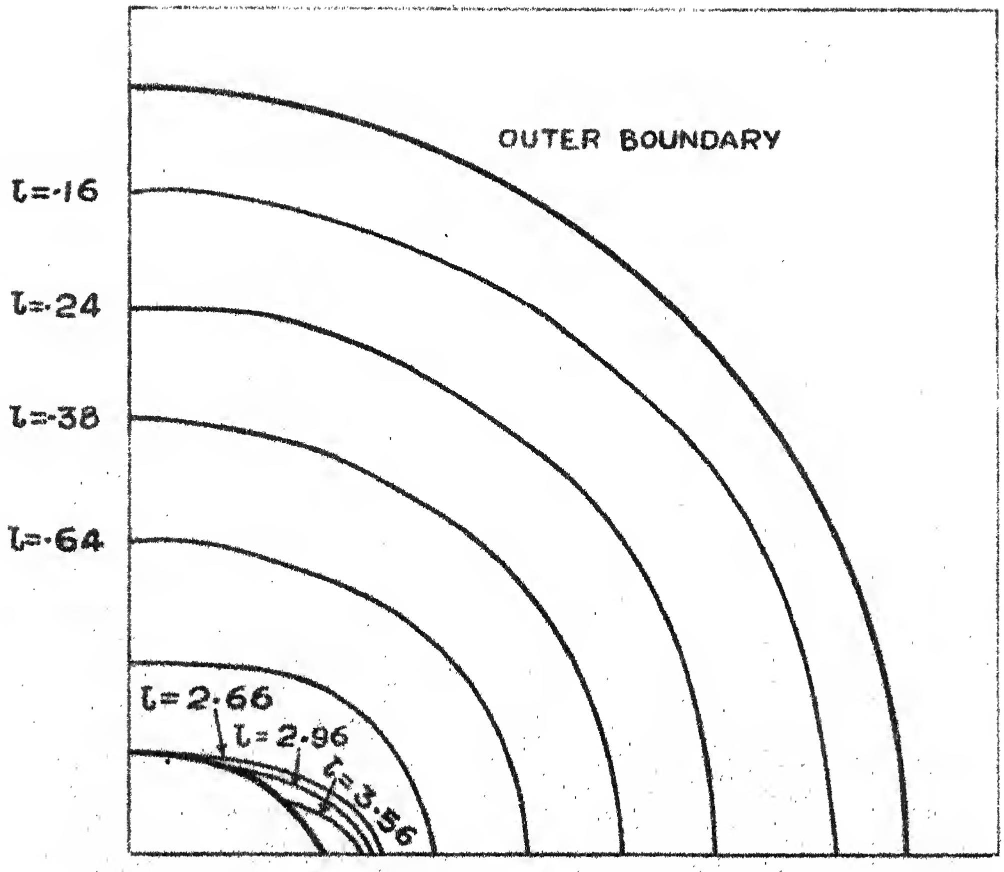
Using (73), (74) and (76), the continuity of normal stress $P_{\xi\xi}$ and shearing stress $P_{\xi\eta}$ at the equilibrium boundary may be seen. The difference in the hoop stress at the equilibrium boundary is given by

$$(P_{\eta\eta})_m - (P_{\eta\eta})_i = \frac{8\mu}{(K+1)(a^2-b^2)} \cdot \frac{\{\varepsilon_1 a^2(1-\cos 2\eta) + \varepsilon_2 b^2(1+\cos 2\eta) - ab(\delta_1+\delta_2)\sin 2\eta\}}{(\cosh 2\xi_0 - \cos 2\eta)} . \tag{79}$$

Some particular cases of great interest may be derived from the results of this problem ; for example if the semi-major and semi-minor axes of the ellipse are

and the semi-major axis a of the ellipse has been given the values 1,2,3. For each value of a , the ratio a/b takes the values 2, 4 and 10. The ratio of maximum shearing stress to $\mu\epsilon/(K+1)$ has been denoted by τ . Nine graphs for these cases are given in the Appendix following this chapter. It is interesting to note that for slender ellipses, the lines of maximum shearing stress in matrix emanating from the boundary of the ellipse resemble to some extent with those of thin rectangular inclusion in a circular region which are given in the Appendix to chapter IV.

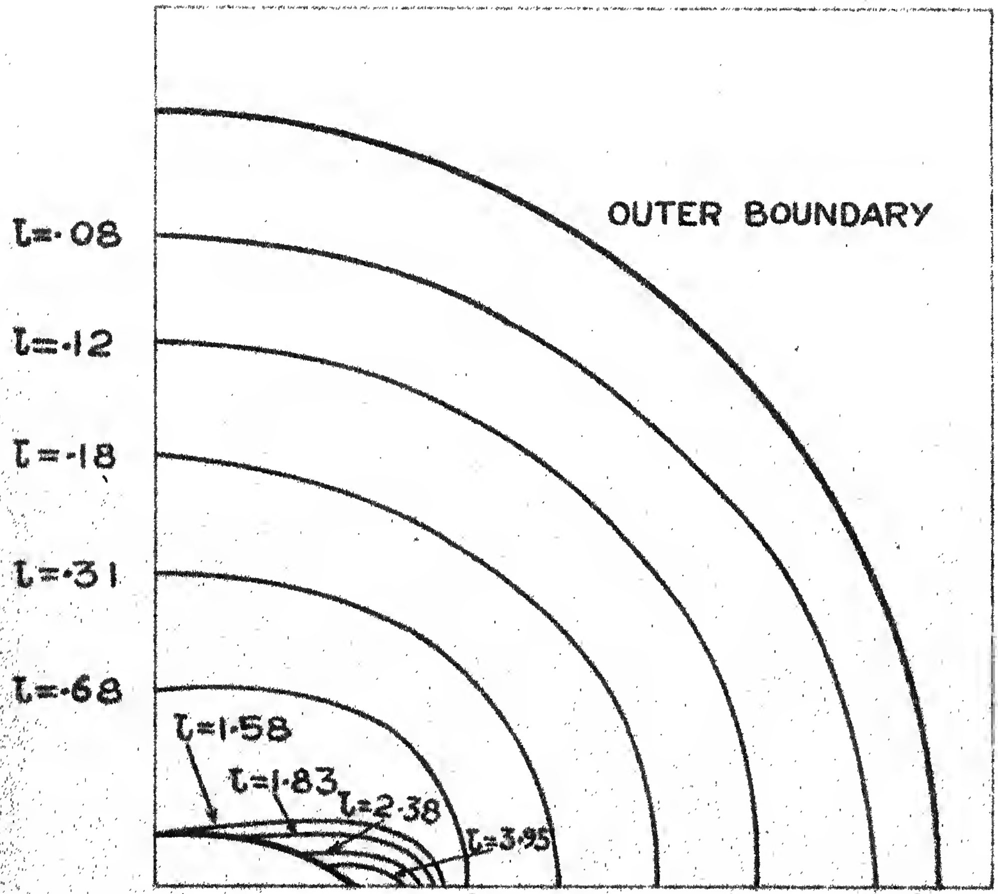
APPENDIX TO CHAPTER III



FIG· 1. LINES OF MAXIMUM SHEARING STRESS FOR AN ELLIPSE

$a=1,\ b=\cdot 5$ AND $\epsilon_1=\epsilon_2=\epsilon$

$\tau=$ MAX· SHEAR STRESS $\big/ \dfrac{\mu\epsilon}{K+1}$



FIG· 2. LINES OF MAXIMUM SHEARING STRESS FOR AN ELLIPSE $a=1$ $b=\cdot 25$ AND $\epsilon_1=\epsilon_2=\epsilon$

$\tau=$ MAX· SHEAR STRESS $\Big/ \dfrac{\mu\epsilon}{K+1}$

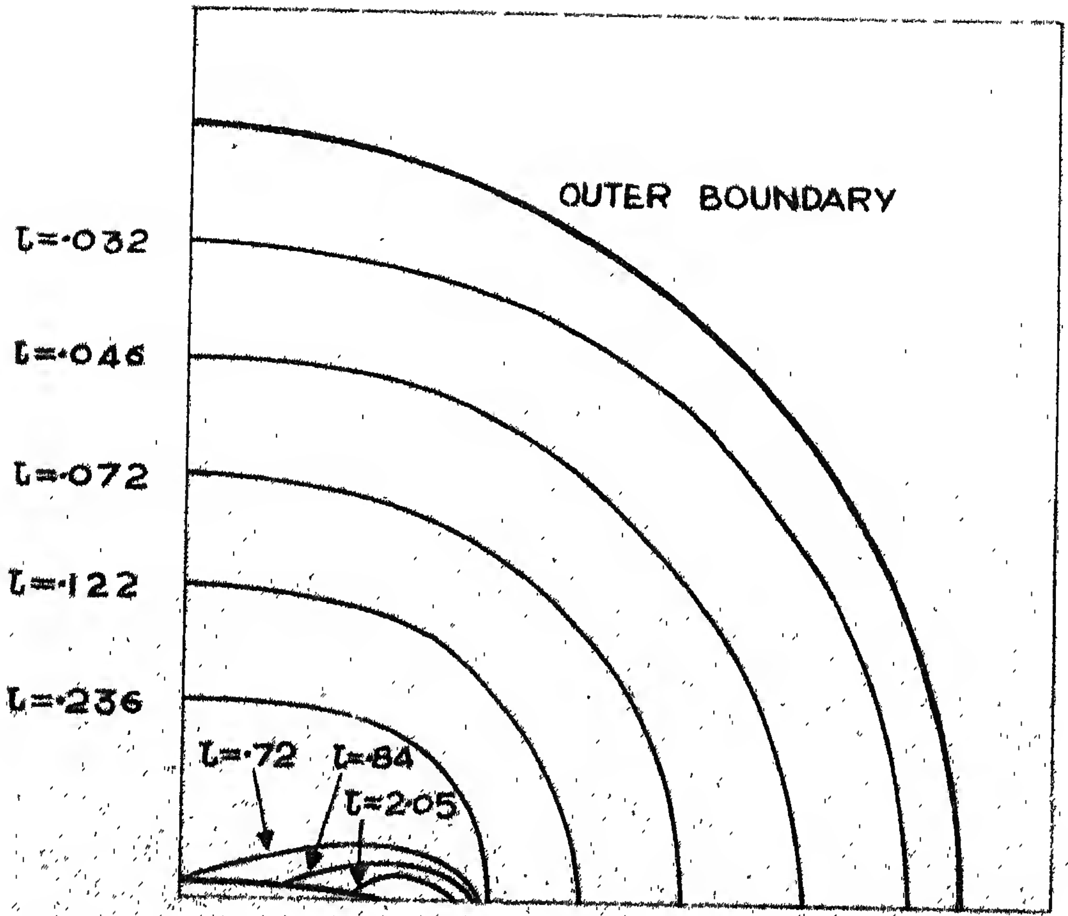


FIG. 3. LINES OF MAXIMUM SHEARING STRESS FOR AN ELLIPSE $a=1, b=\cdot 1$ AND $\epsilon_1=\epsilon_2=\epsilon$

τ = MAX SHEAR STRESS/$\dfrac{(\mu\epsilon)}{(K+1)}$

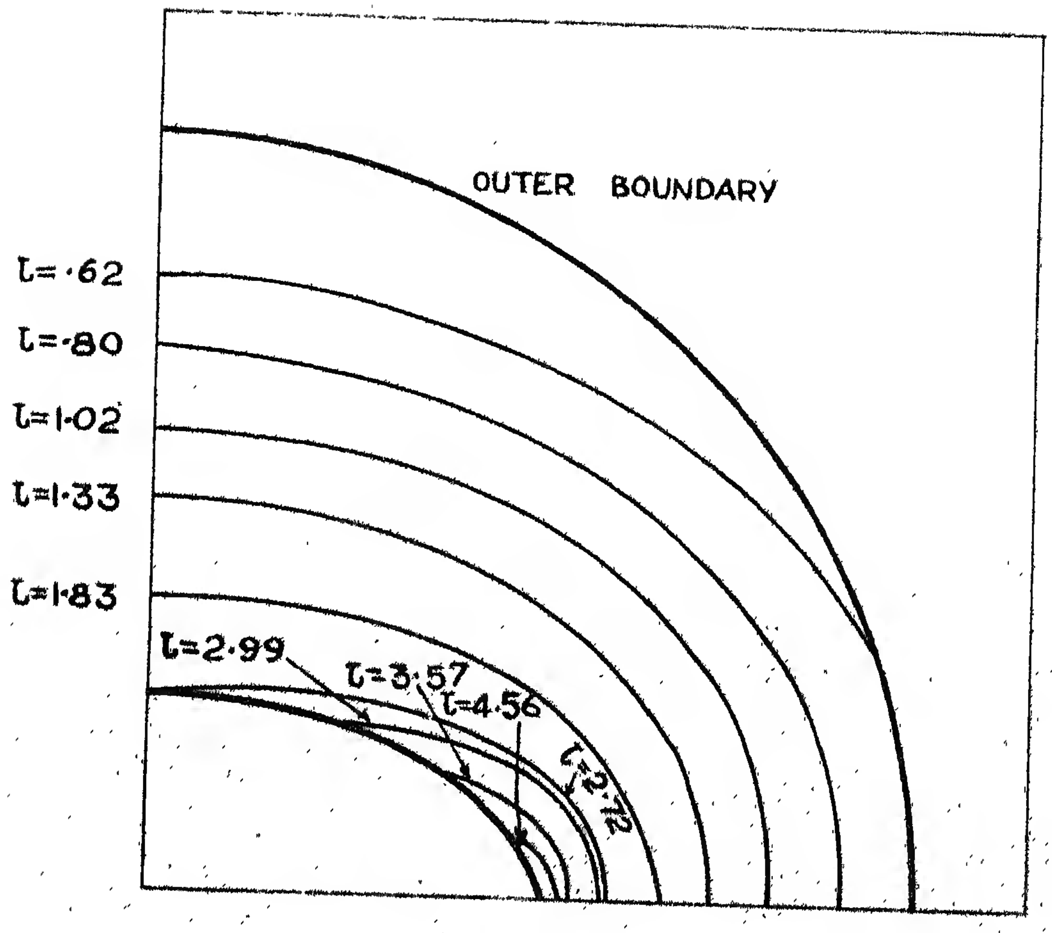
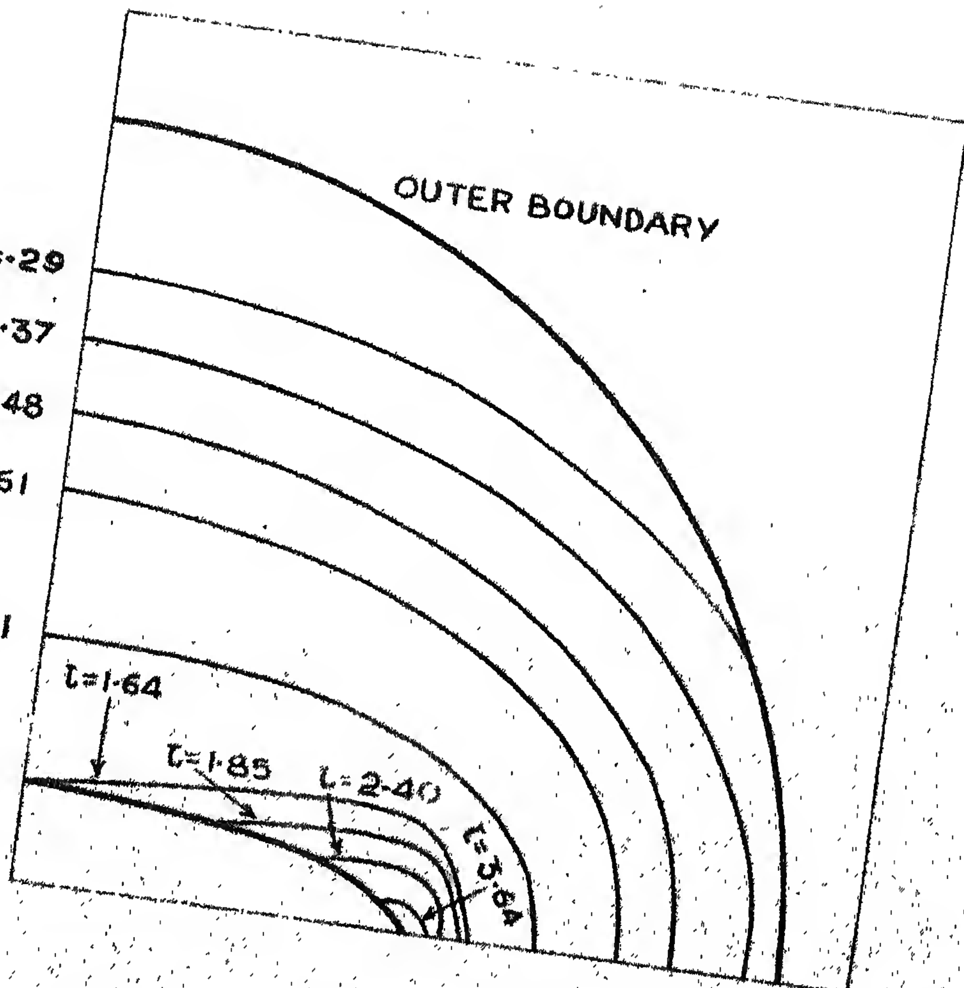
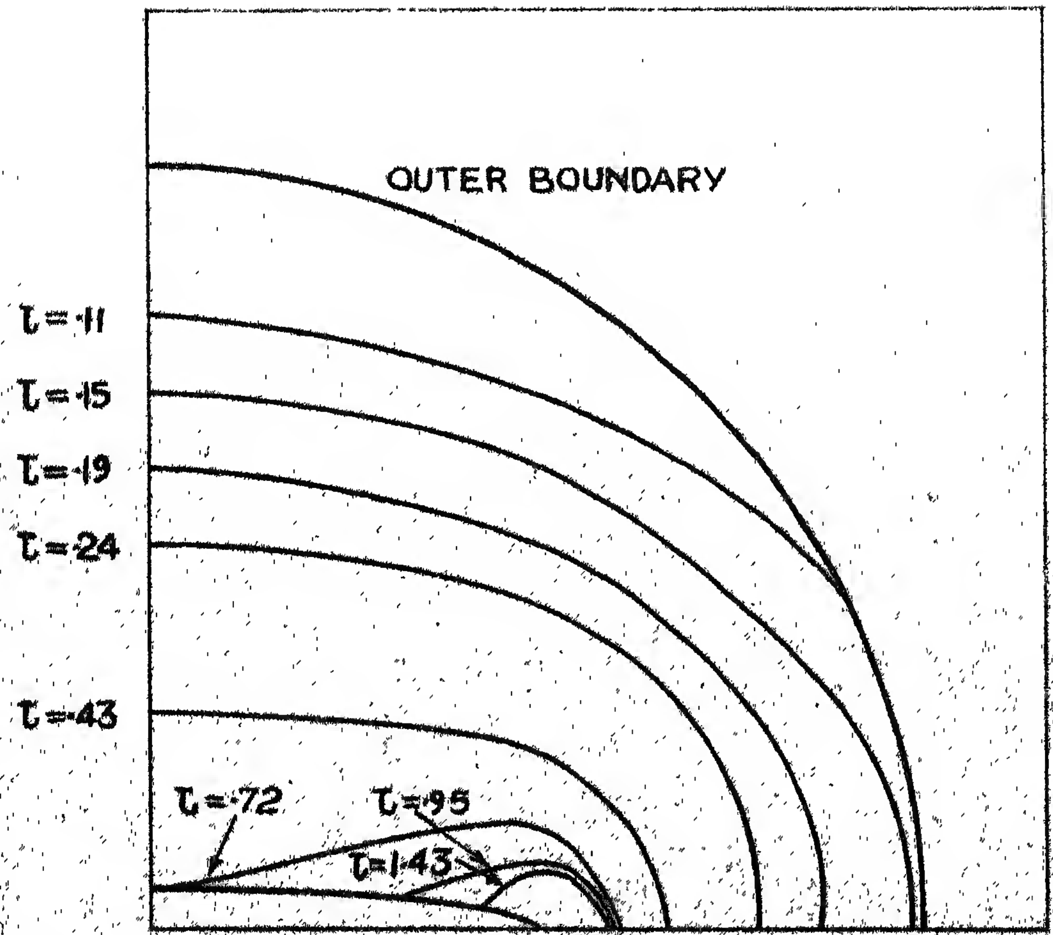


FIG. 4. LINES OF MAXIMUM SHEARING STRESS FOR AN ELLIPSE $a=2$, $b=1$ AND $\epsilon_1=\epsilon_2=\epsilon$

τ = MAX. SHEAR STRESS$/\frac{\mu\epsilon}{\kappa+1}$



FIG·5. LINES OF MAXIMUM SHEARING STRESS FOR AN ELLIPSE
$a=2$, $b=\cdot 5$ AND $\epsilon_1=\epsilon_2=\epsilon$
τ = MAX· SHEAR STRESS/$\frac{\mu\epsilon}{K+1}$



FIG· 6 LINES OF MAXIMUM SHEARING STRESS FOR AN ELLIPSE $a=2, b=\cdot 2$, AND $\epsilon_1=\epsilon_2=\epsilon$.

$\tau=$ MAX· SHEAR STRESS/ $\dfrac{\mu\epsilon}{(K+1)}$

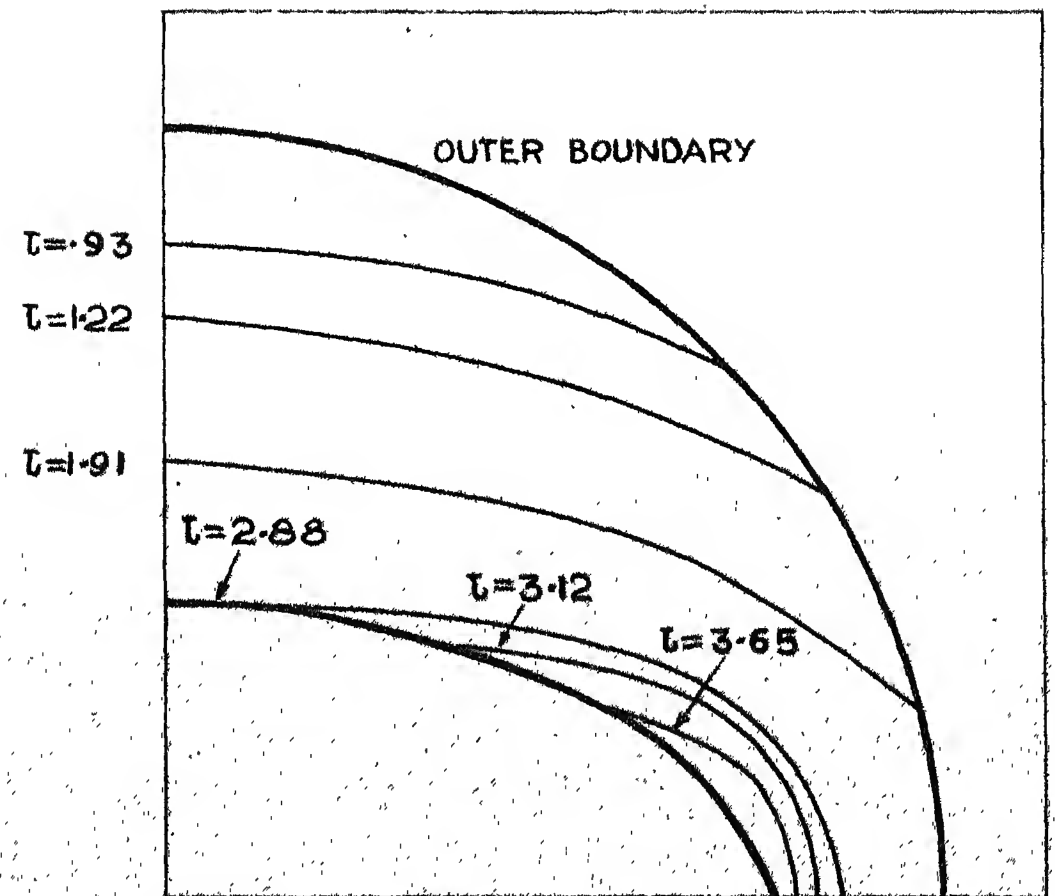


FIG. 7. LINES OF MAXIMUM SHEARING STRESS FOR AN ELLIPSE $a=3$, $b=1.5$ AND $\epsilon_1=\epsilon_2=\epsilon$

τ = MAX SHEAR STRESS $\Big/ \dfrac{\mu\epsilon}{K+1}$

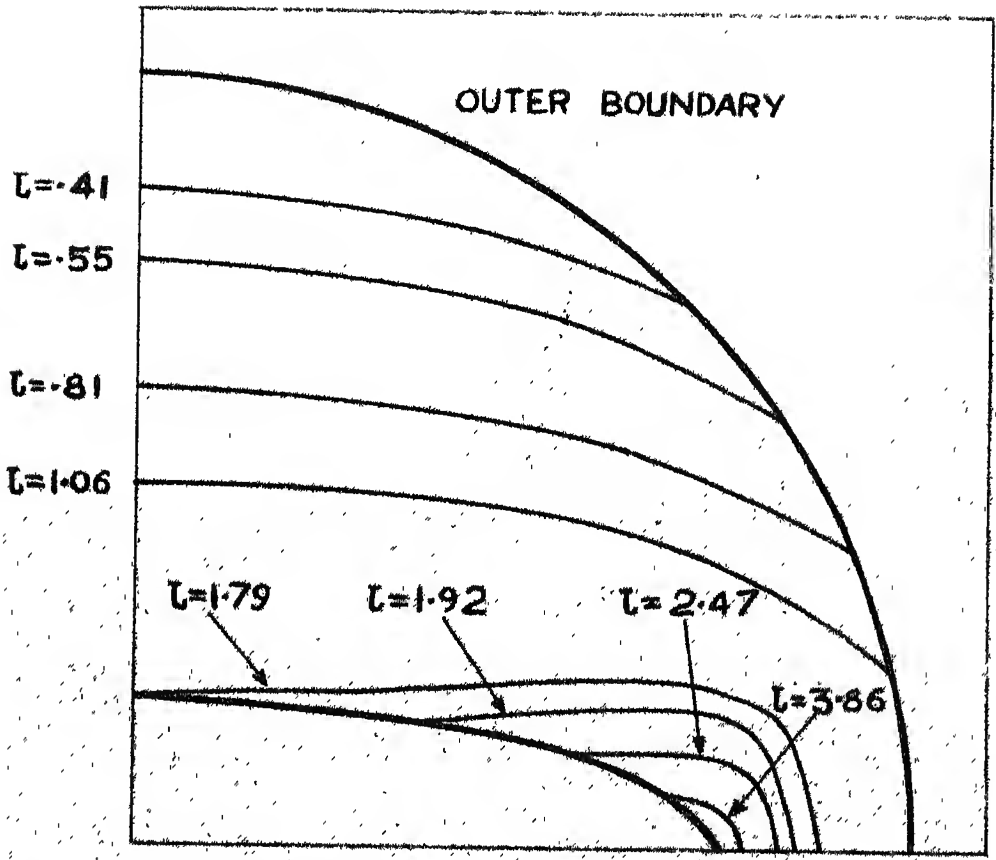
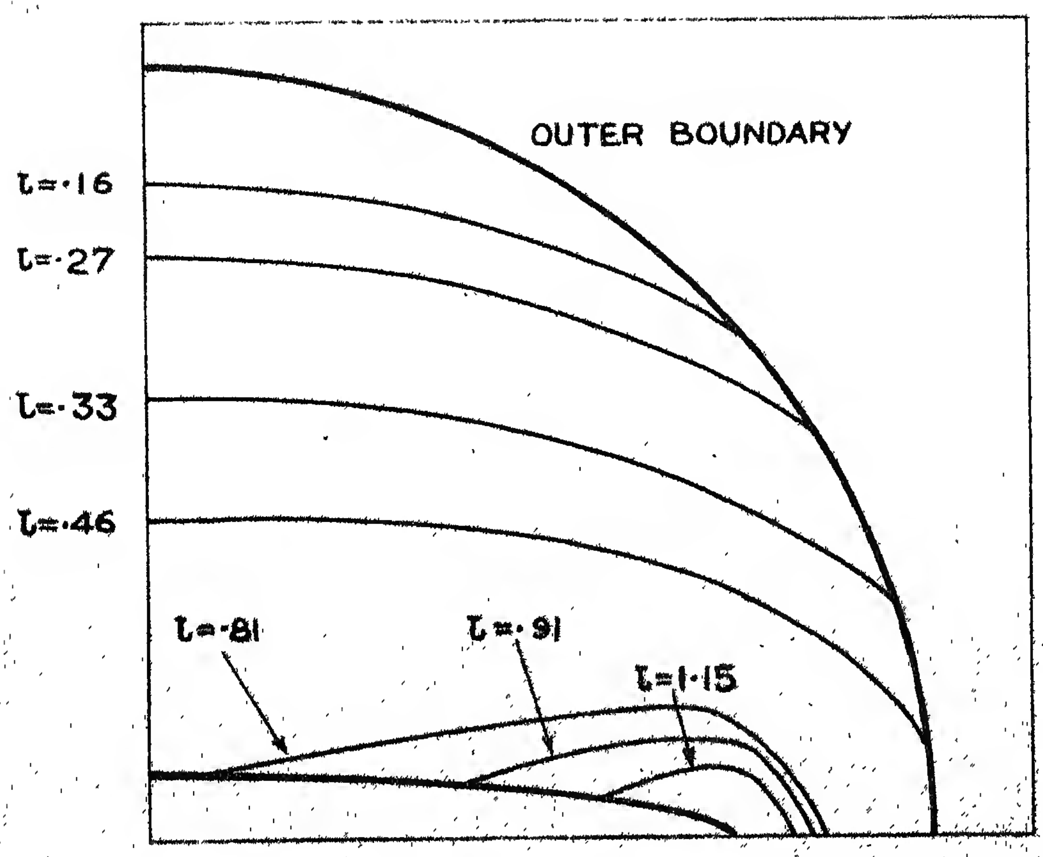


FIG. 8. LINES OF MAXIMUM SHEARING STRESS FOR AN ELLIPSE- $a=3$, $b=.75$ AND $\epsilon_1=\epsilon_2=\epsilon$

$\tau =$ MAX SHEAR STRESS$\Big/\dfrac{\mu\epsilon}{K+1}$



FIG·9. LINES OF MAXIMUM SHEARING STRESS FOR AN ELLIPSE $a=3, b=3$ AND $\epsilon_1=\epsilon_2=\epsilon$

τ = MAX· SHEAR STRESS $/ \frac{\mu\epsilon}{K+1}$

CHAPTER IV

RECTANGULAR INCLUSION IN A CIRCULAR REGION

In the analysis of this chapter, the stress and displacement fields have been evaluated for a rectangular inclusion in a circular region. Consider a circular ring whose outer boundary is a circle of radius R and inner boundary is a concentric rectangle whose sides are $2a$ and $2b$ (Figure 3 p.48). The origin is taken at the centre of the rectangle which is also the centre of the outer circle. The outer boundary of the matrix is denoted by L_o and the inner boundary by L .

The displacement components of the inclusion in Cartesian coordinates in the absence of the matrix are prescribed by $(\epsilon_1 x, \epsilon_2 y)$ with respect to the origin at the centre of L_o.

At the equilibrium interface

$$u_b^+ - u_b^- = -\epsilon_1 x = g_1(t)$$

and $$v_b^+ - v_b^- = -\epsilon_2 y = g_2(t) \,, \tag{81}$$

where u_b^+ , v_b^+ and u_b^- , v_b^- have the same meaning as before and they are measured from their natural states as stated previously.

From (33), (36) and (81), it may be seen that

$$g(t) = -(\epsilon_1 + \epsilon_2)\frac{t}{2} - (\epsilon_1 - \epsilon_2)\frac{\bar{t}}{2} \,,$$

$$h(t) = (\epsilon_1 + \epsilon_2)\,\bar{t} + (\epsilon_1 - \epsilon_2)\frac{t}{2} + (\epsilon_1 - \epsilon_2)\frac{\bar{t}}{2}\frac{\overline{dt}}{dt} \,. \tag{82}$$

Substituting $g(t)$ and $h(t)$ from (82) in (39) and (40) respectively and evaluating the contour integrals, the expressions of $\phi_*(z)$ and $\psi_*(z)$ for inclusion and matrix come out to be

$$\phi_{*i}(z) = -\frac{\mu(\epsilon_1+\epsilon_2)z}{(K+1)} - \frac{\mu(\epsilon_1-\epsilon_2)}{\pi\, i(K+1)}\left[z \log \frac{z^2-(a-ib)^2}{z^2-(a+ib)^2} + a \log \frac{(z-ib)^2-a^2}{(z+ib)^2-a^2} + ib \log \frac{(z-a)^2+b^2}{(z+a)^2+b^2}\right], \tag{83}$$

$$\psi_{*i}(z) = \frac{\mu(\epsilon_1-\epsilon_2)z}{(K+1)} + \frac{2\mu(\epsilon_1+\epsilon_2)}{\pi i(K+1)}\left[z \log \frac{z^2-(a-ib)^2}{z^2-(a+ib)^2}\right.$$

$$\left. + a \log \frac{(z-ib)^2-a^2}{(z+ib)^2-a^2} + ib \log \frac{(z-a)^2+b^2}{(z+a)^2+b^2}\right]$$

$$+ \frac{\mu(\epsilon_1-\epsilon_2)}{\pi i(K+1)}\left[a \log \frac{(z+ib)^2-a^2}{(z-ib)^2-a^2} + ib \log \frac{(z-a)^2+b^2}{(z+a)^2+b^2}\right]$$

and

$$\phi_{*m}(z) = -\frac{\mu(\epsilon_1-\epsilon_2)}{\pi i(K+1)}\left[z \log \frac{z^2-(a-ib)^2}{z^2-(a+ib)^2} + a \log \frac{(z-ib)^2-a^2}{(z+ib)^2-a^2}\right.$$

$$\left. + ib \log \frac{(z-a)^2+b^2}{(z+a)^2+b^2}\right], \tag{84}$$

$$\psi_{*m}(z) = \frac{2\mu(\epsilon_1+\epsilon_2)}{\pi i(K+1)}\left[z \log \frac{z^2-(a-ib)^2}{z^2-(a+ib)^2} + a \log \frac{(z-ib)^2-a^2}{(z+ib)^2-a^2}\right.$$

$$\left. + ib \log \frac{(z-a)^2+b^2}{(z+a)^2+b^2}\right] + \frac{\mu(\epsilon_1-\epsilon_2)}{\pi\, i(K+1)}\left[a \log \frac{(z+ib)^2-a^2}{(z-ib)^2-a^2}\right.$$

$$\left. + ib \log \frac{(z-a)^2+b^2}{(z+a)^2+b^2}\right].$$

The equation of L_o is $|z| = R$. On the outer boundary L_o, the boundary condition is given by (41). In the present case $f(t) = 0$. Using (84), $f_o(t)$ may be found out. This value of $f_o(t)$ is substituted in (41) and the functions $\phi_o(z)$ and $\psi_o(z)$ which are analytic in the region $|z| \leqslant R$ may be determined using the methods described in chapter I. They are

$$\phi_o(z) = -\frac{\mu(\epsilon_1-\epsilon_2)}{\pi i(K+1)}\left[z\log\frac{R^4-(a+ib)^2z^2}{R^4-(a-ib)^2z^2} + a\log\frac{(R^2+ibz)^2-a^2z^2}{(R^2-ibz)^2-a^2z^2}\right.$$

$$\left. + ib\log\frac{(R^2-az)^2+b^2z^2}{(R^2+az)^2+b^2z^2}\right] + \frac{2\mu(\epsilon_1+\epsilon_2)}{\pi i(K+1)}\left[\right.$$

$$\frac{R^2}{z}\log\frac{R^4-(a+ib)^2z^2}{R^4-(a-ib)^2z^2} + a\log\frac{(R^2+ibz)^2-a^2z^2}{(R^2-ibz)^2-a^2z^2}$$

$$\left. - ib\log\frac{(R^2-az)^2+b^2z^2}{(R^2+az)^2+b^2z^2} - \frac{2i\,abz}{R^2}\right], \tag{85}$$

$$\psi_o(z) = \frac{\mu(\epsilon_1-\epsilon_2)}{\pi i(K+1)}\left[a\log\frac{(R^2-ibz)^2-a^2z^2}{(R^2+ibz)^2-a^2z^2} + \right.$$

$$+ ib \log \frac{(R^2-az)^2+b^2z^2}{(R^2+az)^2+b^2z^2} - \frac{2(a+ib)^2 R^2 z}{\{R^4-(a+ib)^2z^2\}}$$

$$+ \frac{2(a-ib)^2 R^2 z}{\{R^4-(a-ib)^2z^2\}} - \frac{2(a+ib)(a-ib)^3 z}{\{R^4-(a-ib)^2z^2\}}$$

$$+ \left.\frac{2(a-ib)(a+ib)^3 z}{\{R^4-(a+ib)^2z^2\}}\right] + \frac{2\mu(\epsilon_1+\epsilon_2)}{\pi i(K+1)}\left[\frac{R^4}{z^3}\log\frac{R^4-(a+ib)^2z^2}{R^4-(a-ib)^2z^2}\right.$$

$$\left. + \frac{4iab}{z}\right].$$

The sectionally holomorphic functions $\phi(z)$ and $\psi(z)$ may now be found out using (37) and (38) and are given below.

$$\phi_i(z) = \phi_o(z) - \frac{\mu(\epsilon_1+\epsilon_2)z}{(K+1)} - \frac{\mu(\epsilon_1-\epsilon_2)}{\pi i(K+1)}\left[z \log \frac{z^2-(a-ib)^2}{z^2-(a+ib)^2}\right.$$

$$\left. + a \log \frac{(z-ib)^2-a^2}{(z+ib)^2-a^2} + ib \log \frac{(z-a)^2+b^2}{(z+a)^2+b^2}\right],$$

(86)

$$\psi_i(z) = \psi_o(z) + \frac{\mu(\epsilon_1-\epsilon_2)z}{(K+1)} + \frac{2\mu(\epsilon_1+\epsilon_2)}{\pi i(K+1)}\left[z \log \frac{z^2-(a-ib)^2}{z^2-(a+ib)^2}\right.$$

$$\left. + a \log \frac{(z-ib)^2-a^2}{(z+ib)^2-a^2} + ib \log \frac{(z-a)^2+b^2}{(z+a)^2+b^2}\right] +$$

$$+\frac{\mu(\epsilon_1-\epsilon_2)}{\pi i(K+1)}\left[a \log \frac{(z+ib)^2-a^2}{(z-ib)^2-a^2}+ib \log \frac{(z-a)^2+b^2}{(z+a)^2+b^2}\right]$$

and

$$\phi_m(z)=\phi_o(z)-\frac{\mu(\epsilon_1-\epsilon_2)}{\pi i(K+1)}\left[z \log \frac{z^2-(a-ib)^2}{z^2-(a+ib)^2}\right.$$

$$\left.+a \log \frac{(z-ib)^2-a^2}{(z+ib)^2-a^2}+ib \log \frac{(z-a)^2+b^2}{(z+a)^2+b^2}\right],$$

(87)

$$\psi_m(z)=\psi_o(z)+\frac{2\mu(\epsilon_1+\epsilon_2)}{\pi i(K+1)}\left[z \log \frac{z^2-(a-ib)^2}{z^2-(a+ib)^2}\right.$$

$$\left.+a \log \frac{(z-ib)^2-a^2}{(z+ib)^2-a^2}+ib \log \frac{(z-a)^2+b^2}{(z+a)^2+b^2}\right]$$

$$+\frac{\mu(\epsilon_1-\epsilon_2)}{\pi i(K+1)}\left[a \log \frac{(z+ib)^2-a^2}{(z-ib)^2-a^2}+ib \log \frac{(z-a)^2+b^2}{(z+a)^2+b^2}\right].$$

Stresses in Cartesian coordinate system are given by (9). Using (87) and (9), it may be seen that

$$(P_{xx}+P_{yy})_i=-\frac{4\mu(\epsilon_1+\epsilon_2)}{(K+1)}-\frac{2\mu(\epsilon_1-\epsilon_2)}{\pi i(K+1)}\left[\log \frac{R^4-z^2(a+ib)^2}{R^4-(a-ib)^2z^2}\right.$$

$$+\frac{2R^4}{\{R^4-(a-ib)^2z^2\}}-\frac{2R^4}{\{R^4-(a+ib)^2z^2\}}-\frac{2R^2(a^2+b^2)}{\{R^4-(a-ib)^2z^2\}}+$$

$$+ \frac{2R^2(a^2+b^2)}{\{R^4-(a+ib)^2\ z^2\}} + \log \frac{z^2-(a-ib)^2}{z^2-(a+ib)^2}$$

$$-\log \frac{R^4-(a-ib)^2\ \bar{z}^2}{R^4-(a+ib)^2\ \bar{z}^2} - \frac{2R^4}{\{R^4-(a+ib)^2\ \bar{z}^2\}}$$

$$+ \frac{2R^4}{\{R^4-(a-ib)^2\ \bar{z}^2\}} + \frac{2R^2(a^2+b^2)}{\{R^4-(a+ib)^2\bar{z}^2\}} - \frac{2R^2(a^2+b^2)}{\{R^4-(a-ib)^2\bar{z}^2\}}$$

$$\left. - \log \frac{\bar{z}^2-(a+ib)^2}{\bar{z}^2-(a-ib)^2} \right] + \frac{4\mu(\epsilon_1+\epsilon_2)}{\pi i(K+1)} \left[- \frac{R^2}{z^2} \log \frac{R^4-(a+ib)^2 z^2}{R^4-(a-ib)^2 z^2} \right.$$

$$\left. - \frac{4iab}{R^2} + \frac{R^2}{\bar{z}^2} \log \frac{R^4-(a-ib)^2\ \bar{z}^2}{R^4-(a+ib)^2\ \bar{z}^2} \right],$$

(88)

$$(P_{xx}+P_{yy})_m = (P_{xx}+P_{yy})_i + \frac{4\mu(\epsilon_1+\epsilon_2)}{(K+1)}$$

and

$$(P_{yy}-P_{xx}+2iP_{xy})_i = - \frac{4\mu(\epsilon_1-\epsilon_2)}{\pi i(K+1)} \left[- \frac{(a+ib)^2 z\ \bar{z}}{\{R^4-(a+ib)^2 z^2\}} \right.$$

$$+ \frac{(a-ib)^2 z\ \bar{z}}{\{R^4-(a-ib)^2 z^2\}} - \frac{2R^4(a+ib)^2 z\bar{z}}{\{R^4-(a+ib)^2 z^2\}^2} +$$

$$+\frac{2R^4(a-ib)^2 z\bar{z}}{\{R^4-(a-ib)^2 z^2\}^2} - \frac{2R^2(a^2+b^2)(a-ib)^2 z\bar{z}}{\{R^4-(a-ib)^2 z^2\}^2}$$

$$+\frac{2R^2(a^2+b^2)(a+ib)^2 z\bar{z}}{\{R^4-(a+ib)^2 z^2\}^2} + \frac{z\,\bar{z}}{z^2-(a-ib)^2}$$

$$-\frac{z\,\bar{z}}{z^2-(a+ib)^2} - \frac{2R^6(a+ib)^2}{\{R^4-(a+ib)^2 z^2\}^2} + \frac{2R^6(a-ib)^2}{\{R^4-(a-ib)^2 z^2\}^2}$$

$$-\frac{(a^2+b^2)(a-ib)^2\{R^4+(a-ib)^2 z^2\}}{\{R^4-(a-ib)^2 z^2\}^2}$$

$$+\frac{(a^2+b^2)(a+ib)^2\{R^4+(a+ib)^2 z^2\}}{\{R^4-(a+ib)^2 z^2\}^2} + \frac{(a^2+b^2)}{z^2-(a-ib)^2}$$

$$\left.-\frac{(a^2+b^2)}{z^2-(a+ib)^2}\right] + \frac{4\mu(\epsilon_1+\epsilon_2)}{\pi i\ (K+1)}\left[\frac{2R^2\bar{z}}{z^3}\log\frac{R^4-(a+ib)^2 z^2}{R^4-(a-ib)^2 z^2}\right.$$

$$+\frac{2R^2(a+ib)^2\bar{z}}{z\{R^4-(a+ib)^2 z^2\}} - \frac{2R^2(a-ib)^2\,\bar{z}}{z\{R^4-(a-ib)^2 z^2\}}$$

$$-\frac{3R^4}{z^4}\log\frac{R^4-(a+ib)^2 z^2}{R^4-(a-ib)^2 z^2} - \frac{4\ iab}{z^2}$$

$$-\frac{2R^4(a+ib)^2}{z^2\{R^4-(a+ib)^2 z^2\}} + \frac{2\ R^4(a-ib)^2}{z^2\{R^4-(a-ib)^2 z^2\}} +$$

$$+ \quad \log \frac{z^2-(a-ib)^2}{z^2-(a+ib)^2} \Bigg] + \frac{2\mu(\epsilon_1-\epsilon_2)}{(K+1)} ,$$

(89)

$$(P_{yy}-P_{xx}+2i\,P_{xy})_m = (P_{yy}-P_{xx}+2iP_{xy})_i - \frac{2\mu(\epsilon_1-\epsilon_2)}{(K+1)} .$$

The displacement components in Cartesian coordinates are given by (10). For inclusion and matrix respectively, they are given by the expressions

$$2\mu(u+iv)_i = -\frac{\mu(\epsilon_1+\epsilon_2)\,\bar{z}}{(K+1)} + \frac{\mu(1-K)(\epsilon_1+\epsilon_2)z}{(K+1)}$$

$$- \frac{\mu(\epsilon_1-\epsilon_2)}{\pi i(K+1)} \Bigg[Kz \log \frac{R^4-(a+ib)^2z^2}{R^4-(a-ib)^2z^2}$$

$$+ aK \log \frac{(R^2+ibz)^2-a^2z^2}{(R^2-ibz)^2-a^2z^2} + ibK \log \frac{(R^2-az)^2+b^2z^2}{(R^2+az)^2+b^2z^2}$$

$$+ Kz \log \frac{z^2-(a-ib)^2}{z^2-(a+ib)^2} + aK \log \frac{(z-ib)^2-a^2}{(z+ib)^2-a^2}$$

$$+ ibK\log \frac{(z-a)^2+b^2}{(z+a)^2+b^2} + z \log \frac{R^4-(a-ib)^2\,\bar{z}^2}{R^4-(a+ib)^2\,\bar{z}^2}$$

$$- \frac{2R^4 z}{R^4-(a-ib)^2\bar{z}^2} + \frac{2R^4 z}{R^4-(a+ib)^2\,\bar{z}^2}$$

$$+ \frac{2R^2(a^2+b^2)z}{R^4-(a-ib)^2\,\bar{z}^2} - \frac{2R^2(a^2+b^2)\;z}{R^4-(a+ib)^2\,\bar{z}^2}$$

$$+ z \log \frac{\bar{z}^2 - (a+ib)^2}{\bar{z}^2 - (a-ib)^2} - a \log \frac{(R^2+ib\bar{z})^2 - a^2\bar{z}^2}{(R^2-ib\bar{z})^2-a^2\,\bar{z}^2}$$

$$+ ib \log \frac{(R^2-a\bar{z})^2+ b^2\,\bar{z}^2}{(R^2+a\bar{z})^2+ b^2\,\bar{z}^2} + \frac{2(a-ib)^2R^2\,\bar{z}}{R^4-(a-ib)^2\,\bar{z}^2}$$

$$- \frac{2(a+ib)^2\,R^2\bar{z}}{R^4-(a+ib)^2\,\bar{z}^2} + \frac{2(a-ib)(a+ib)^3\,\bar{z}}{R^4-(a+ib)^2\,\bar{z}^2}$$

$$- \frac{2(a^2+b^2)(a-ib)^2\bar{z}}{R^4-(a-ib)^2\,\bar{z}^2} - a \log \frac{(\bar{z}- ib)^2- a^2}{(\bar{z} +ib)^2- a^2}$$

$$+ ib \log \left.\frac{(\bar{z} -a)^2+b^2}{(\bar{z} +a)^2+b^2}\right] + \frac{2\mu(\epsilon_1+\epsilon_2)}{\pi i\;(K+1)}\left[\frac{KR^2}{z} \log \frac{R^4-(a+ib)^2z^2}{R^4-(a-ib)^2z^2}\right.$$

$$+aK\log \frac{(R^2+ibz)^2-a^2z^2}{(R^2-ibz)^2-a^2z^2} - ibK \log \frac{(R^2-az)^2+ b^2z^2}{(R^2+az)^2+ b^2z^2} +$$

$$+ 2(1-K)iab \frac{z}{R^2} - \frac{R^2 z}{\bar{z}^2} \log \frac{R^4-(a-ib)^2 \bar{z}^2}{R^4-(a+ib)^2 \bar{z}^2}$$

$$+ \frac{R^4}{\bar{z}^3} \log \frac{R^4-(a-ib)^2 \bar{z}^2}{R^4-(a+ib)^2 \bar{z}^2} - \frac{4iab}{\bar{z}} + \bar{z} \log \frac{\bar{z}^2-(a+ib)^2}{\bar{z}^2-(a-ib)^2}$$

$$+ a \log \frac{(\bar{z}+ib)^2-a^2}{(\bar{z}-ib)^2-a^2} - ib \log \frac{(\bar{z}-a)^2+b^2}{(\bar{z}+a)^2+b^2} \Big]$$

and (90)

$$2\mu(u+iv)_m = 2\mu(u+iv)_i + \frac{\mu(\varepsilon_1-\varepsilon_2)}{(K+1)} \bar{z} - \frac{\mu(1-K)(\varepsilon_1+\varepsilon_2)z}{(K+1)} .$$

It may be verified from the expressions of stresses in (88) and (89), that the normal stress and shearing stress vanish on L_o i.e. on $|z| = R$, which they should. The continuity of normal and shearing stresses may also be seen, provided the proper branches of logarithmic functions as given below are chosen.

Let θ_1, θ_2, θ_3 and θ_4 be the angles as shown in Figure 4 p. 48, then for matrix

$$I\left(\log \frac{z+a-ib}{z-a-ib}\right) = \theta_1 ,$$

$$I\left(\log \frac{z+a-ib}{z+a+ib}\right) = \theta_2 ,$$

$$I\left(\log \frac{z-a+ib}{z+a+ib}\right) = \theta_3 ,$$

$$I\left(\log \frac{z-a+ib}{z-a-ib}\right) = \theta_4 ,$$

and for inclusion

$$I\left(\log \frac{z+a-ib}{z-a-ib}\right) = \theta_1 ,$$

$$I\left(\log \frac{z+a-ib}{z+a+ib}\right) = \theta_2 ,$$

$$I\left(\log \frac{z-a+ib}{z+a+ib}\right) = \theta_3 ,$$

$$I\left(\log \frac{z-a+ib}{z-a-ib}\right) = \theta_4 ,$$

where I means the imaginary part of a complex quantity. It may be noted that the orientations of θ_1, θ_2, θ_3 and θ_4 for inclusion are different from matrix.

In the above analysis if the radius of the circle $R \to \infty$, then the results for rectangular inclusion in an infinite medium are obtained ((9)). Further, if $a = b$, then the results of square inclusion in a circular inclusion are obtained. Also if $a = b$ and $R \to \infty$, the results of square inclusion in an infinite medium are obtained.

Numerical work has been done for this problem and is reported in the form of graphs in Appendix following this

chapter. Lines of constant maximum shearing stress have been drawn in these graphs for the cases when $\epsilon_1 = \epsilon_2 = \epsilon$ and $\epsilon_1 = -\epsilon_2 = \epsilon$. For both the cases i.e. $\epsilon_1 = \epsilon_2 = \epsilon$ and $\epsilon_1 = -\epsilon_2 = \epsilon$, square inclusions for $a = 1$ and $a = 2$ and rectangular inclusions of sides $a = 1$ and $b = .1$ and $a = 1$ and $b = .5$ have been considered. The radius of the outer circle is taken as 4 in all the cases mentioned above. Because of the symmetry only a quadrant of a circle is considered. It may be noted that the boundary conditions at the outer as well as inner boundary were verified numerically also so as to have a check on the numerical computations.

APPENDIX TO CHAPTER IV

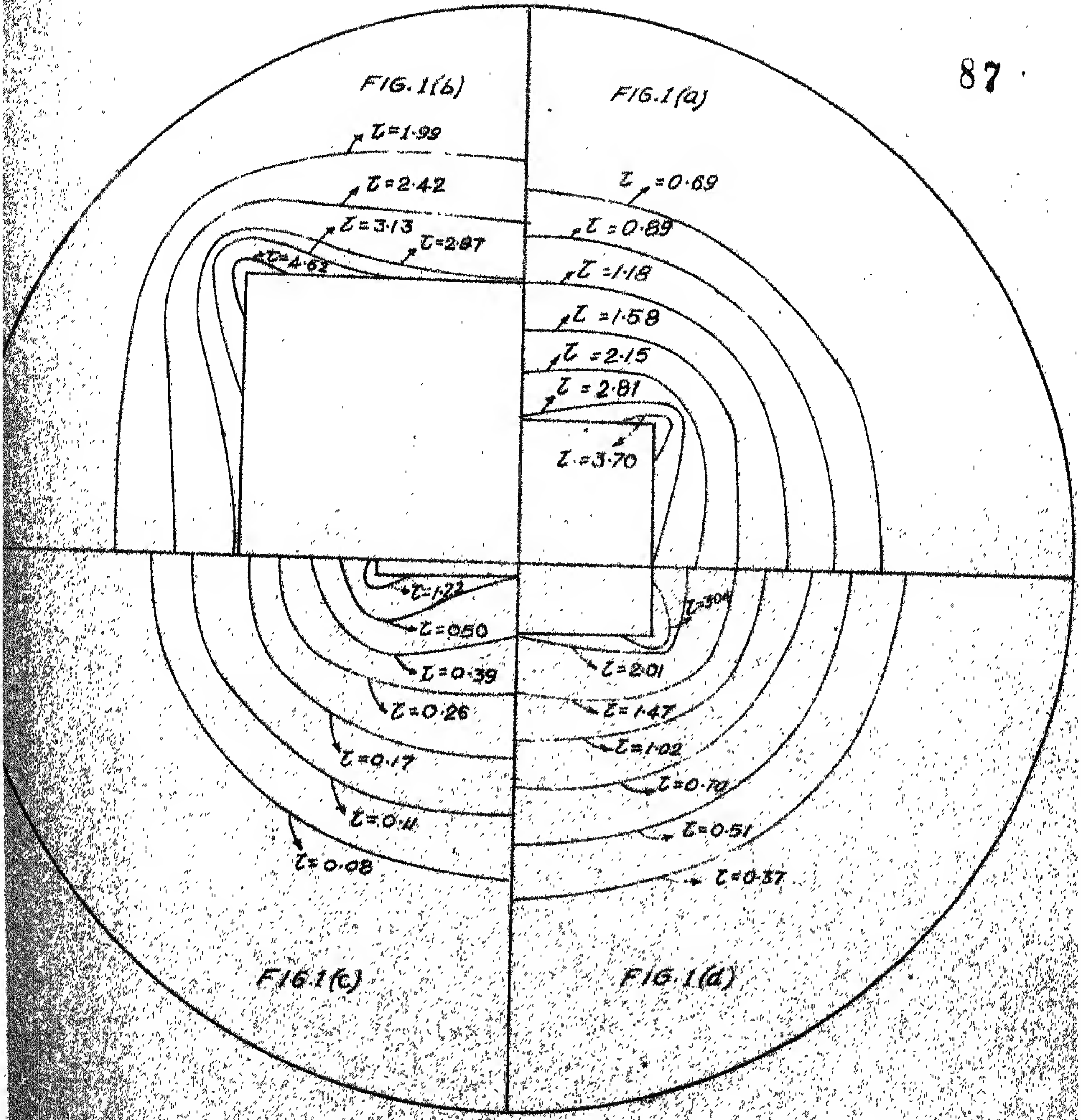


FIG. 1-a, b, c, d

LINES OF MAXIMUM SHEARING STRESS FOR (a) A SQUARE OF SIDE 1 (b) A SQUARE OF SIDE 2 (c) A RECTANGLE OF SIDES 1, 1 (d) A RECTANGLE OF SIDES 1, .5 IN A CIRCULAR DISC OF RADIUS 4, AND ϵ_1 = ϵ_2 = ϵ. FIGURES SHOW ONLY A QUADRANT IN EACH CASE.

τ = MAX. SHEARING STRESS/ $\frac{\mu \epsilon}{(\kappa \; 1)}$

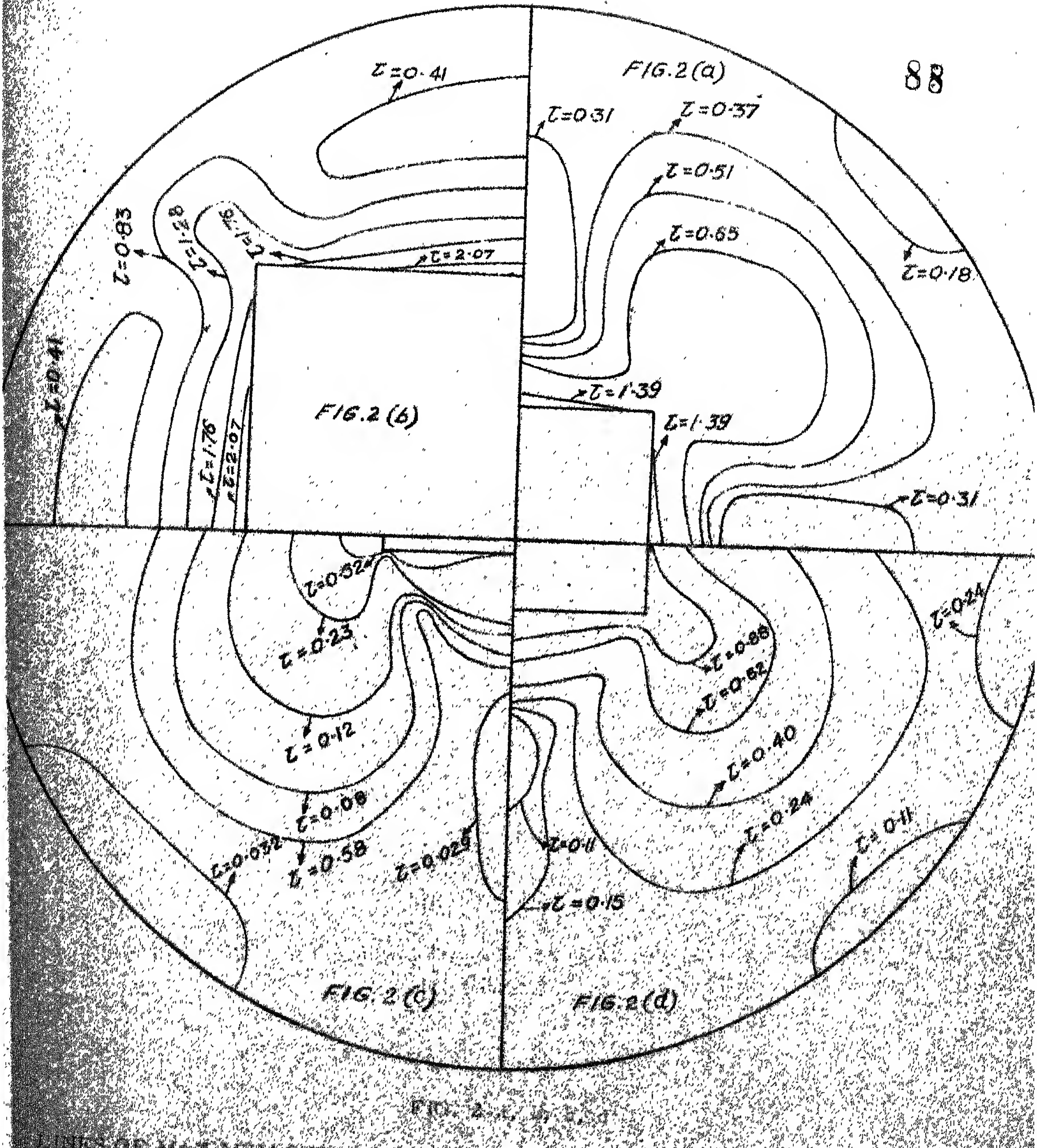


FIG. 2 [illegible]

LINES OF MAXIMUM SHEARING STRESS FOR [illegible] A SQUARE OF SIDE 1 [illegible] A SQUARE OF SIDE [illegible] A RECTANGLE OF SIDES 1, 1 [illegible] A RECTANGLE OF SIDES 1, .5 IN A CIRCULAR DISC OF RADIUS [illegible] AND [illegible] = ϵ_2 = ϵ. FIGURES SHOW ONLY A QUADRANT IN EACH CASE.

τ = MAX. SHEARING STRESS [illegible]

P A R T   B

CHAPTER V

THREE-DIMENSIONAL AND TWO-DIMENSIONAL POINT FORCE PROBLEMS

In the previous chapters, inclusion problems when the matrix is finite, have been solved with the help of the Hilbert theorem. However, when the matrix is infinite, energy methods ((10)) and point force methods ((11)) have also been applied. Energy methods ((12, 13)) require considerable guessing of the equilibrium interface which is not possible except in some simple cases. Using point force technique, a systematic study for ellipsoidal inclusion was made by Eshelby ((14)) and the two-dimensional complex variable formalism was first given by Jaswon and Bhargava ((15)). The point force technique consists of the following hypothetical operations.

First, cut out the inclusion from the remaining material and allow it to undergo freely the non-elastic deformation. Next apply the surface tractions to the inclusion which bring it to its original size. Replace the stressed inclusion in the cavity from which it was cut out. Rejoin the material across the cut. Apply a distribution of point forces on the inclusion boundary so as to neutralise the effect of aforesaid applied surface tractions. This brings us to the required configuration. Thus, as it were, the effect of the deforming inclusion is to bring into play a layer of point forces along its boundary. Therefore, if the elastic fields due to a single force in an infinite medium are known, the corresponding fields for inclusion problem would be obtained by the integrated effects of the whole distribution of point forces generated along the equilibrium boundary.

Subsequent chapters therefore deal with the point force, or concentrated force as it is some times called. It may however be remarked that point force finds applications not only in inclusion problems, but also in material testing, soil mechanics etc.

In this chapter, the Love's solution ((16)) of the equations of equilibrium for axi-symmetric case in terms of two functions is described; secondly the displacement

components for a point force in three-dimensional infinite medium and stresses due to a point force in two-dimensional region are given. For details of Love's solution, one might see chapter (XI) of Love's book ((16)). We give, first some important results of Love's solution which would be needed in subsequent chapters for solving point force problems. Some theory is also given to make the thesis self-contained.

The equations of equilibrium for the axi-symmetric case (variation with respect to θ is zero) in cylindrical coordinates (R, θ, Z) $(x = R\cos\theta,\ y = R\sin\theta\ ,\ z = Z)$ are

$$\frac{\partial P_{RR}}{\partial R} + \frac{\partial P_{RZ}}{\partial Z} + \frac{P_{RR} - P_{\theta\theta}}{R} = 0, \qquad (91)$$

and
$$\frac{\partial P_{RZ}}{\partial R} + \frac{\partial P_{ZZ}}{\partial Z} + \frac{P_{RZ}}{R} = 0\ . \qquad (92)$$

The displacement component u_θ in the θ direction is identically zero in this case and the displacement components u_R and u_Z in the direction of R and Z are related to the strain components by the equations

$$e_{RR} = \frac{\partial u_R}{\partial R}\ ,$$

$$e_{\theta\theta} = \frac{u_R}{R} ,$$

$$e_{ZZ} = \frac{\partial u_Z}{\partial Z} ,$$

$$e_{RZ} = \frac{1}{2} \left(\frac{\partial u_R}{\partial Z} + \frac{\partial u_Z}{\partial R} \right) , \tag{93}$$

$$e_{R\theta} = 0 ,$$

$$e_{\theta Z} = 0 .$$

Let $$P_{RZ} = - \frac{\partial^2 \phi}{\partial R \partial Z} , \tag{94}$$

where ϕ is a function of R and Z only.

It may be seen from (92) that

$$P_{ZZ} = \frac{\partial^2 \phi}{\partial R^2} + \frac{1}{R} \frac{\partial \phi}{\partial R} ; \tag{95}$$

no function of R is added, because any such function can be included in ϕ .

From the expressions of e_{RR} and $e_{\theta\theta}$ given in (93), it is clear that

$$e_{RR} = \frac{\partial}{\partial R} (R \, e_{\theta\theta}) . \tag{96}$$

With the help of stress-strain relations

$$P_{RR} = \lambda \Delta + 2 \mu e_{RR} ,$$

$$
\begin{aligned}
P_{\theta\theta} &= \lambda\Delta + 2\mu e_{\theta\theta} \ , \\
P_{ZZ} &= \lambda\Delta + 2\mu e_{ZZ} \ , \\
P_{R\theta} &= 2\mu e_{R\theta} \ , \\
P_{RZ} &= 2\mu e_{RZ} \ , \\
P_{\theta Z} &= 2\mu e_{\theta Z} \ ,
\end{aligned} \tag{97}
$$

where $\Delta = (e_{RR} + e_{\theta\theta} + e_{ZZ})$ and λ and μ are the Lamé constants; the equation (96) may be written in terms of stresses as

$$
P_{RR} - \nu(P_{\theta\theta} + P_{ZZ}) = \frac{\partial}{\partial R}\{R(P_{\theta\theta} - \nu P_{RR} - \nu P_{ZZ})\} .
$$

whence,

$$
(1+\nu)\ (P_{RR} - P_{\theta\theta}) = R\frac{\partial}{\partial R}(P_{\theta\theta} - \nu P_{RR} - \nu P_{ZZ}) . \tag{98}
$$

Let a new function $F(R,Z)$ be introduced which is given by the equation

$$
P_{RR} = \frac{\partial^2\phi}{\partial Z^2} + F , \tag{99}
$$

then the equation (91) can be written as

$$
(1+\nu)\frac{\partial F}{\partial R} + \frac{\partial}{\partial R}(P_{\theta\theta} - \nu P_{RR} - \nu P_{ZZ}) = 0 . \tag{100}
$$

It may be seen from (100) that

$$
P_{\theta\theta} = \nu\nabla^2\phi - F ; \tag{101}
$$

no arbitrary function of Z need be added, for any such function may be included in ϕ, ∇^{1} denotes

$$\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2},$$

the subjects of operation being independent of θ.

All the non zero stress-components have now been expressed in terms of the two functions ϕ and F. From the fact that the sum of the principal stresses is a harmonic function, it can be derived that ϕ satisfies the biharmonic equation

$$\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2}\right)\left(\frac{\partial^2\phi}{\partial R^2} + \frac{1}{R}\frac{\partial\phi}{\partial R} + \frac{\partial^2\phi}{\partial Z^2}\right) = 0. \tag{102}$$

The functions ϕ and F are not independent of each other and the relation between them may be derived as follows:

The equation $u_R/R = e_{\theta\theta}$ can be written as

$$u_R = \frac{R}{E}\left\{P_{\theta\theta} - \nu\,(P_{RR} + P_{ZZ})\right\},$$

where E is the Young's modulus of elasticity.

Also from (95), (99) and (101)

$$u_R = -\frac{(1+\nu)}{E} R F;$$

and then the equation $P_{RZ} = 2\mu e_{RZ}$ can be written

$$\frac{\partial u_Z}{\partial R} = - \frac{2(1+\nu)}{E} \frac{\partial^2 \phi}{\partial R \partial Z} + \frac{(1+\nu)}{E} R \frac{\partial F}{\partial Z} . \qquad (103)$$

Also the equation $e_{ZZ} = \frac{1}{E}\left\{ P_{ZZ} - \nu (P_{RR} + P_{\theta\theta}) \right\}$ may be written as

$$\frac{\partial u_Z}{\partial Z} = \frac{(1+\nu)}{E} \left\{ \frac{\partial^2 \phi}{\partial R^2} + \frac{1}{R} \frac{\partial \phi}{\partial R} - \nu \nabla^2 \phi \right\} . \qquad (104)$$

The equations (103) and (104) are compatible if

$$(1-\nu) \frac{\partial}{\partial R}(\nabla^2 \phi) + \frac{\partial^3 \phi}{\partial R \partial Z^2} = R \frac{\partial^2 F}{\partial Z^2} . \qquad (105)$$

If a new function Ω is introduced by the equation

$$RF = \frac{\partial \Omega}{\partial R} + \frac{\partial \phi}{\partial R} , \qquad (106)$$

then from (105), it may be seen that

$$\frac{\partial^2 \Omega}{\partial Z^2} = (1-\nu) \nabla^2 \phi , \qquad (107)$$

where , as before, no arbitrary function of Z need be added.

The stress components are thus expressible in terms of the functions ϕ and Ω as

$$P_{RR} = \frac{\partial^2\phi}{\partial Z^2} + \frac{1}{R}\left(\frac{\partial\phi}{\partial R} + \frac{\partial\Omega}{\partial R}\right),$$

$$P_{\theta\theta} = \nu\nabla^2\phi - \frac{1}{R}\left(\frac{\partial\phi}{\partial R} + \frac{\partial\Omega}{\partial R}\right),$$

$$P_{RZ} = -\frac{\partial^2\phi}{\partial R\partial Z},$$

$$P_{ZZ} = \frac{\partial^2\phi}{\partial R^2} + \frac{1}{R}\frac{\partial\phi}{\partial R}, \tag{108}$$

$$P_{R\theta} = 0, \quad P_{\theta Z} = 0;$$

when Ω is introduced in (103) and (104), they may be written as

$$\frac{\partial u_Z}{\partial R} = \frac{(1+\nu)}{E}\frac{\partial}{\partial R}\left(\frac{\partial\Omega}{\partial Z} - \frac{\partial\phi}{\partial Z}\right)$$

and $$\frac{\partial u_z}{\partial Z} = \frac{(1+\nu)}{E}\frac{\partial}{\partial Z}\left(\frac{\partial\Omega}{\partial Z} - \frac{\partial\phi}{\partial Z}\right).$$

Hence

$$\begin{aligned} u_R &= -\frac{(1+\nu)}{E}\left(\frac{\partial\Omega}{\partial R} + \frac{\partial\phi}{\partial R}\right), \\ u_Z &= \frac{(1+\nu)}{E}\left(\frac{\partial\Omega}{\partial Z} - \frac{\partial\phi}{\partial Z}\right). \end{aligned} \tag{109}$$

The value of Δ can be found out in two different ways.

Firstly $\Delta = e_{RR} + e_{\theta\theta} + e_{ZZ}$

$$= \frac{\partial u_R}{\partial R} + \frac{u_R}{R} + \frac{\partial u_Z}{\partial Z} \tag{110}$$

Also

$$= \frac{(1+\nu)}{E}\left[(1-2\nu)\nabla^2\phi - \nabla^2\Omega\right] .$$

$$\Delta = \frac{(1-2\nu)}{E}\Theta = \frac{(1-2\nu)(1+\nu)}{E}\nabla^2\phi \quad , \tag{111}$$

where Θ is the sum of three principal stresses.

$$\Theta = P_{RR} + P_{\theta\theta} + P_{ZZ} .$$

It follows from (110) and (111) that besides (107), Ω also satisfies the equation $\nabla^2\Omega = 0$.

Thus

$$\nabla^2\Omega = 0$$

and

$$\frac{\partial^2\Omega}{\partial z^2} = (1-\nu)\nabla^2\phi \quad . \tag{112}$$

This completes the Love's solution of the equations of equilibrium for the axisymmetric case in terms of the functions ϕ and Ω.

We now give the displacement components (u_x, u_y, u_z) due to a point force (X_o, Y_o, Z_o) acting at a point (x_1, y_1, z_1) in a three-dimensional infinite medium.

$$(u_x, u_y, u_z) = \frac{(\lambda+3\mu)}{8\pi\mu(\lambda+2\mu)}\left(\frac{X_o}{r_1}, \frac{Y_o}{r_1}, \frac{Z_o}{r_1}\right) + \frac{(\lambda+\mu)}{8\pi\mu(\lambda+2\mu)}\left(\frac{x-x_1}{r_1}, \frac{y-y_1}{r_1}, \frac{z-z_1}{r_1}\right)\left\{\frac{X_o(x-x_1)+Y_o(y-y_1)+Z_o(z-z_1)}{r_1^2}\right\} \tag{113}$$

$$P_{xy} = -\frac{X_o}{4\pi(1-\nu)}\left[(1-2\nu)\frac{(y-k)}{\{(x-h)^2+(y-k)^2\}}+\frac{2(x-h)^2(y-k)}{\{(x-h)^2+(y-k)^2\}^2}\right]$$

$$-\frac{Y_o}{4\pi(1-\nu)}\left[(1-2\nu)\frac{(x-h)}{\{(x-h)^2+(y-k)^2\}}+\frac{2(x-h)(y-k)^2}{\{(x-h)^2+(y-k)^2\}^2}\right]. \tag{114}$$

The above stresses can be derived from the complex potentials given in Green and Zerna ((1)) or from Muskhelishvili ((2)).

CHAPTER VI

ELASTIC CONE UNDER AN AXIAL FORCE

The solution to the problem of a point force acting along the axis of a cone at its vertex is known ((16)). In the present problem, the axial force is assumed to act at any point on the axis of the cone. This problem is physically very interesting. The solution is obtained by the following operation :

Imagine a point force $(0,0, Z_o)$ acting at a point $(0,0, d)$ in an infinite elastic medium. This states that a point force of magnitude Z_o acts along the Z-axis at a point d from the origin. The expressions for displacements are known ((16)) and are given by the equation (113); from where the stresses everywhere in the elastic medium can be found out. This we call the first stress system. Now suppose that there is a cone

whose vertex coincides with the origin and the axis with the Z-axis. On the curved surface of the cone, we can find the stresses from the first stress system. Particularly, we can find the normal and shearing stresses on the curved surface of the cone.

We now solve another elastic problem by applying a second system of forces on the surface of the cone. This second system of forces are obtained from the normal and shearing stresses opposite in sign to those obtained from the first stress system on the surface of the cone. The two stress systems in the cone are then superposed. The solution to the problem when the curved surface is free from external forces and the cone is subjected to an axial point force, is then obtained.

Consider a semi-infinite elastic cone with vertex at the origin and the Z-axis coinciding with the axis of the cone. The semi-vertical angle of the cone is θ_o. Let a force of magnitude Z_o act at a point (0, 0, d) along the axis (Figure 1 p. 139).

The displacement components (u_x, u_y, u_z) in Cartesian coordinates for the first system of stresses when the medium is supposed to be infinite are obtained from (113) by putting x_1 and y_1 to be zero and so also X_o, Y_o. They are

$$u_x = \frac{(\lambda+\mu)Z_o}{8\pi\mu(\lambda+2\mu)} \frac{x(z-d)}{r_2^3},$$

$$u_y = \frac{(\lambda+\mu)Z_o}{8\pi\mu(\lambda+2\mu)} \frac{y(z-d)}{r_2^3}, \tag{115}$$

$$u_z = \frac{(\lambda+\mu)Z_o}{8\pi\mu(\lambda+2\mu)} \frac{(z-d)^2}{r_2^3} + \frac{(\lambda+3\mu)Z_o}{8\pi\mu(\lambda+2\mu)} \frac{1}{r_2},$$

where $r_2^2 = x^2 + y^2 + (z-d)^2$.

We use spherical polar coordinates (r, θ, φ) so that $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$ $(0 \leqslant r < \infty,$ $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi < 2\pi)$. It may be noted that θ in cylindrical coordinates is different from θ in spherical polar coordinates. The displacement components in (115) may be written in spherical polar coordinates as

$$u_r = \frac{(\lambda+\mu)Z_o}{8\pi\mu(\lambda+2\mu)} \frac{(rp-d)(r-dp)}{(r^2+d^2-2drp)^{3/2}} + \frac{(\lambda+3\mu)Z_o}{8\pi\mu(\lambda+2\mu)} \frac{p}{(r^2+d^2-2drp)^{1/2}},$$

$$u_\theta = \frac{(\lambda+\mu)Z_o}{8\pi\mu(\lambda+2\mu)} \frac{dq(rp-d)}{(r^2+d^2-2drp)^{3/2}} - \frac{(\lambda+3\mu)Z_o}{8\pi\mu(\lambda+2\mu)} \frac{q}{(r^2+d^2-2drp)^{1/2}},$$

$$\tag{116}$$

$$u_\varphi = 0,$$

where $p = \cos\theta$ and $q = \sin\theta$.

The corresponding stresses are

$$(P_{rr})_1 = -\frac{\mu\ Z_o}{4\pi(\lambda+2\mu)}\ \frac{rp}{(r^2+d^2-2drp)^{3/2}}$$

$$+\frac{\mu\ Z_o}{4\pi(\lambda+2\mu)}\ \frac{d(2p^2-1)}{(r^2+d^2-2drp)^{3/2}}$$

$$-\frac{3(\lambda+\mu)\ Z_o}{4\pi(\lambda+2\mu)}\ \frac{(rp-d)(r-dp)^2}{(r^2+d^2-2drp)^{5/2}}\ ,$$

$$(P_{r\theta})_1 = \frac{\mu\ Z_o}{4\pi(\lambda+2\mu)}\ \frac{q(r-2dp)}{(r^2+d^2-2drp)^{3/2}}$$

$$-\frac{3(\lambda+\mu)\ Z_o}{4\pi(\lambda+2\mu)}\ \frac{\{dpqr^2-d^2q(1+p^2)r+d^3pq\}}{(r^2+d^2-2drp)^{5/2}}\ ,$$

$$(P_{\theta\theta})_1 = \frac{\mu\ Z_o}{4\pi(\lambda+2\mu)}\ \frac{\{pr+d(2q^2-1)\}}{(r^2+d^2-2drp)^{3/2}}$$

$$+\frac{3(\lambda+\mu)\ Z_o}{4\pi(\lambda+2\mu)}\ \frac{d^2q^2(d-rp)}{(r^2+d^2-2drp)^{5/2}}\ ,$$

$$(P_{\varphi\varphi})_1 = \frac{\mu\ Z_o}{4\pi(\lambda+2\mu)}\ \frac{(rp-d)}{(r^2+d^2-2drp)^{3/2}}\ , \qquad (117)$$

$$(P_{r\varphi})_1 = 0 \quad \text{and} \quad (P_{\theta\varphi})_1 = 0.$$

The subscript 1 is used to the above stresses to indicate that the stresses are due to the 'first'

stress system.

We now apply a second system of forces on the surface of the cone which are obtained from the normal stress $(P_{\theta\theta})_2$ and shearing stresses $(P_{r\theta})_2$ and $(P_{\theta\varphi})_2$ where ,

$$(P_{\theta\theta})_2 = -(P_{\theta\theta})_1\ ,\ (P_{r\theta})_2 = -(P_{r\theta})_1 \text{ and } (P_{\theta\varphi})_2 = -(P_{\theta\varphi})_1\ ,$$

on the surface of the cone.

Hence

$$\left\{(P_{\theta\theta})_2\right\}_{\theta=\theta_0} = -\left[\frac{\mu\ Z_0}{4\pi(\lambda+2\mu)}\ \frac{\{p_0 r + d(2q_0^2-1)\}}{(r^2+d^2-2drp_0)^{3/2}} + \frac{3(\lambda+\mu)\ Z_0}{4\pi(\lambda+2\mu)}\ \frac{d^2q_0^2\ (d-rp_0)}{(r^2+d^2-2drp_0)^{5/2}}\right],$$

$$\left\{(P_{r\theta})_2\right\}_{\theta=\theta_0} = -\left[\frac{\mu\ Z_0}{4\pi(\lambda+2\mu)}\ \frac{q_0(r-2dp_0)}{(r^2+d^2-2drp_0)^{3/2}} - \frac{3(\lambda+\mu)\ Z_0}{4\pi(\lambda+2\mu)}\ \frac{\{dp_0q_0r^2 - d^2q_0(1+p_0^2)r + d^3p_0q_0\}}{(r^2+d^2-2drp_0)^{5/2}}\right],$$

$$(P_{\theta\varphi})_2 = 0\ ;\ \text{where } p_0 = \cos\theta_0 \text{ and } q_0 = \sin\theta_0. \tag{118}$$

The mathematics required for this auxiliary problem is not trivial and is interesting. Some details of

calculation are given in the next chapter from which the solution has been built up. At this stage it suffices to mention that the second system of stresses

$$(P_{rr})_2\ ,\ (P_{\theta\theta})_2\ ,\ (P_{\varphi\varphi})_2\ ,\ (P_{r\theta})_2\ ,\ (P_{r\varphi})_2 \text{ and } (P_{\theta\varphi})_2$$

come out to be

$$(P_{rr})_2 = -\frac{(\lambda+\mu)Z_o}{4\pi(\lambda+2\mu)d^{1/2}r^{3/2}}\int_0^\infty \frac{E_{11}\cos m'\alpha - E_{22}\sin m'\alpha}{E_{oo}(1+4\alpha^2)\cosh \pi\alpha}\,d\alpha,$$

$$(P_{\theta\theta})_2 = -\frac{(\lambda+\mu)Z_o}{4\pi(\lambda+2\mu)d^{1/2}r^{3/2}}\int_0^\infty \frac{E_{33}\cos m'\alpha - E_{44}\sin m'\alpha}{E_{oo}(1+4\alpha^2)\cosh \pi\alpha}\,d\alpha,$$

$$(P_{\varphi\varphi})_2 = -\frac{(\lambda+\mu)Z_o}{4\pi(\lambda+2\mu)d^{1/2}r^{3/2}}\int_0^\infty \frac{E_{55}\cos m'\alpha - E_{66}\sin m'\alpha}{E_{oo}(1+4\alpha^2)\cosh \pi\alpha}\,d\alpha,$$

$$(P_{r\theta})_2 = \frac{(\lambda+\mu)Z_o q}{4\pi(\lambda+2\mu)d^{1/2}r^{3/2}}\int_0^\infty \frac{E_{77}\cos m'\alpha - E_{88}\sin m'\alpha}{E_{oo}(1+4\alpha^2)\cosh \pi\alpha}\,d\alpha,$$

$$(P_{r\varphi})_2 = 0 \quad \text{and} \quad (P_{\theta\varphi})_2 = 0\ , \tag{119}$$

where $m' = \log(d/r)$, α is the variable of integration and the quantities E_{oo}, E_{11}, E_{22}, E_{33}, E_{44}, E_{55}, E_{66}, E_{77} and E_{88} are given below.

$$E_{oo} = p_o(\tfrac{1}{4} + \alpha^2)^2 P_{-\frac{1}{2}+i\alpha}(p_o)\, P_{-\frac{1}{2}+i\alpha}(p_o) - q_o^2(\tfrac{1}{4} + \alpha^2) P'_{-\frac{1}{2}+i\alpha}(p_o) P_{-\frac{1}{2}+i\alpha}(p_o)$$

$$- p_o\{2(1-\nu) + q_o^2\,(\tfrac{1}{4} + \alpha^2)\}\, P'_{-\frac{1}{2}+i\alpha}(p_o)\, P'_{-\frac{1}{2}+i\alpha}(p_o) \quad ,$$

where in the above expression and the expressions below,

$P_{-\frac{1}{2}+i\alpha}(\cos\theta)$ is the Legendre function of first kind and

$P'_{-\frac{1}{2}+i\alpha}(\cos\theta)$ is its derivative with respect to the argument in the bracket;

$$E_{11} = H_1 - 2\alpha\, H_2,$$

$$E_{22} = H_2 + 2\alpha\, H_1,$$

$$E_{33} = H_3 - 2\alpha\, H_4,$$

$$E_{44} = H_4 + 2\alpha\, H_3,$$

$$E_{55} = H_5 - 2\alpha\, H_6,$$

$$E_{66} = H_6 + 2\alpha\, H_5 \ ,$$

$$E_{77} = H_7 - 2\alpha\, H_8 \ ,$$

$$E_{88} = H_8 + 2\alpha\, H_7 \ ,$$

where

$$H_1 = -\left\{q^2(1-2\nu)\, P'_{-\frac{1}{2}+i\alpha}(p) + \frac{3p}{2}(\tfrac{1}{4} + \alpha^2)\, P_{-\frac{1}{2}+i\alpha}(p)\right\} E_1 +$$

$$+ \alpha \left\{ 2q^2 P'_{-\frac{1}{2}+i\alpha}(p) + p\left(\frac{1}{4}+\alpha^2\right) P_{-\frac{1}{2}+i\alpha}(p) \right\} E_2$$

$$+ \left\{ q^2 P'_{-\frac{1}{2}+i\alpha}(p) - 2p\left(\frac{1}{4}+\alpha^2\right) P_{-\frac{1}{2}+i\alpha}(p) \right\} E_3$$

$$- 2\alpha q^2 P'_{-\frac{1}{2}+i\alpha}(p) \cdot E_4 ,$$

$$H_2 = - \left\{ q^2(1-2\nu) P'_{-\frac{1}{2}+i\alpha}(p) + \frac{3p}{2}\left(\frac{1}{4}+\alpha^2\right) P_{-\frac{1}{2}+i\alpha}(p) \right\} E_2$$

$$- \alpha \left\{ 2q^2 P'_{-\frac{1}{2}+i\alpha}(p) + p\left(\frac{1}{4}+\alpha^2\right) P_{-\frac{1}{2}+i\alpha}(p) \right\} E_1$$

$$+ \left\{ q^2 P'_{-\frac{1}{2}+i\alpha}(p) - 2p\left(\frac{1}{4}+\alpha^2\right) P_{-\frac{1}{2}+i\alpha}(p) \right\} E_4$$

$$+ 2\alpha q^2 P'_{-\frac{1}{2}+i\alpha}(p) \cdot E_3 ,$$

$$H_3 = \left\{ p^2(-2.5+2\nu) P'_{-\frac{1}{2}+i\alpha}(p) - \frac{3p}{2}\left(\frac{1}{4}+\alpha^2\right) P_{-\frac{1}{2}+i\alpha}(p) \right\} E_1 +$$

$$- \alpha \left\{ p^2 P'_{-\frac{1}{2}+i\alpha}(p) + p\left(\frac{1}{4}+\alpha^2\right) P_{-\frac{1}{2}+i\alpha}(p) \right\} E_2$$

$$+ \left\{ 2p\left(\frac{1}{4}+\alpha^2\right) P_{-\frac{1}{2}+i\alpha}(p) + (q^2+2p^2) P'_{-\frac{1}{2}+i\alpha}(p) \right\} E_3 +$$

$$+ 2\,\alpha\, q^2\, P'_{-\frac{1}{2}+i\alpha}(p)\cdot E_4\,,$$

$$H_4 = \left\{p^2\,(-2.5 + 2\nu)\,P'_{-\frac{1}{2}+i\alpha}(p) - \frac{3p}{2}\left(\frac{1}{4}+\alpha^2\right)P_{-\frac{1}{2}+i\alpha}(p)\right\}E_2$$

$$+ \alpha\left\{p^2\,P'_{-\frac{1}{2}+i\alpha}(p) + p\left(\frac{1}{4}+\alpha^2\right)P_{-\frac{1}{2}+i\alpha}(p)\right\}E_1$$

$$+ \left\{2p\left(\frac{1}{4}+\alpha^2\right)P_{-\frac{1}{2}+i\alpha}(p) + (2p^2+q^2)\,P'_{-\frac{1}{2}+i\alpha}(p)\right\}E_4$$

$$- 2\,\alpha\, q^2\, P'_{-\frac{1}{2}+i\alpha}(p)\cdot E_3\,,$$

$$H_5 = \left\{\left(-\frac{p^2}{2} + 2 - 2\nu + \nu q^2\right)P'_{-\frac{1}{2}+i\alpha}(p) + p(1-2\nu)\left(\frac{1}{4}+\alpha^2\right)P_{-\frac{1}{2}+i\alpha}(p)\right\}E_1$$

$$+ \alpha\,(p^2 + 2\nu q^2)\,P'_{-\frac{1}{2}+i\alpha}(p)\cdot E_2 - 2\,P'_{-\frac{1}{2}+i\alpha}(p)\cdot E_3\,,$$

$$H_6 = \left\{\left(\frac{p^2}{2} + 2 - 2\nu + \nu q^2\right)P'_{-\frac{1}{2}+i\alpha}(p) + p(1-2\nu)\left(\frac{1}{4}+\alpha^2\right)P_{-\frac{1}{2}+i\alpha}(p)\right\}E_2$$

$$- \alpha(p^2 + 2\nu q^2)\,P'_{-\frac{1}{2}+i\alpha}(p)\cdot E_1 - 2\,P'_{-\frac{1}{2}+i\alpha}(p)\cdot E_4\,,$$

$$H_7 = \left[\left\{2(1-\nu) + \frac{1}{4} + \alpha^2\right\}p\,P'_{-\frac{1}{2}+i\alpha}(p) + \left(\frac{1}{4}+\alpha^2\right)P_{-\frac{1}{2}+i\alpha}(p)\right]E_1 -$$

$$- \left\{ p\, P'_{-\frac{1}{2}+i\alpha}(p) + 2\left(\frac{1}{4}+\alpha^2\right) P_{-\frac{1}{2}+i\alpha}(p) \right\} E_3$$

$$+ 2p\,\alpha\, P'_{-\frac{1}{2}+i\alpha}(p) \cdot E_4 ,$$

$$H_8 = \left[\left\{ 2(1-\nu) + \frac{1}{4} + \alpha^2 \right\} p\, P'_{-\frac{1}{2}+i\alpha}(p) + \left(\frac{1}{4}+\alpha^2\right) P_{-\frac{1}{2}+i\alpha}(p) \right] E_2$$

$$- \left\{ p\, P'_{-\frac{1}{2}+i\alpha}(p) + 2\left(\frac{1}{4}+\alpha^2\right) P_{-\frac{1}{2}+i\alpha}(p) \right\} E_4$$

$$- 2p\,\alpha\, P'_{-\frac{1}{2}+i\alpha}(p) \cdot E_3 ,$$

$$E_1 = - 2p_o \left\{ 2(1-\nu) + \left(\frac{1}{4}+\alpha^2\right) q_o^2 \right\} P'_{-\frac{1}{2}+i\alpha}(p_o)\, P'_{-\frac{1}{2}+i\alpha}(-p_o)$$

$$- \left(\frac{1}{4}+\alpha^2\right) \left\{ 4(1-\nu) + q_o^2 \right\} P'_{-\frac{1}{2}+i\alpha}(-p_o)\, P_{-\frac{1}{2}+i\alpha}(p_o)$$

$$+ \left(\frac{1}{4}+\alpha^2\right) \left\{ 4(1-\nu) - q_o^2 \right\} P'_{-\frac{1}{2}+i\alpha}(p_o)\, P_{-\frac{1}{2}+i\alpha}(p_o)$$

$$+ 2p_o \left(\frac{1}{4}+\alpha^2\right)^2 P_{-\frac{1}{2}+i\alpha}(p_o)\, P_{-\frac{1}{2}+i\alpha}(-p_o),$$

$$E_2 = 2\alpha\left(\frac{1}{4}+\alpha^2\right) q_o^2 \left\{ P'_{-\frac{1}{2}+i\alpha}(-p_o)\, P_{-\frac{1}{2}+i\alpha}(p_o) - \right.$$

$$- P'_{-\frac{1}{2}+i\alpha}(p_o)\, P_{-\frac{1}{2}+i\alpha}(-p_o)\Big\} ,$$

$$E_3 = -p_o\Big\{4(1-\nu)^2+1-\nu+(\tfrac{1}{4}+\alpha^2)q_o^2(\tfrac{5}{2}-2\nu)\Big\}P'_{-\frac{1}{2}+i\alpha}(p_o)\,.$$

$$.P'_{-\frac{1}{2}+i\alpha}(-p_o) +3(1-\nu)p_o^2(\tfrac{1}{4}+\alpha^2)P'_{-\frac{1}{2}+i\alpha}(p_o)\,.$$

$$.P_{-\frac{1}{2}+i\alpha}(-p_o) - (\tfrac{1}{4}+\alpha^2)\Big\{\tfrac{3}{2}+(1-2\nu)(1+\tfrac{1}{2}p_o^2)\Big\}\,.$$

$$.\,P'_{-\frac{1}{2}+i\alpha}(-p_o)\,P_{-\frac{1}{2}+i\alpha}(p_o)+p_o(\tfrac{1}{4}+\alpha^2)^2(\tfrac{5}{2}-2\nu)\,.$$

$$.\,P_{-\frac{1}{2}+i\alpha}(p_o)\,P_{-\frac{1}{2}+i\alpha}(-p_o) ,$$

$$E_4 = \alpha\Big[p_o\Big\{2(1-\nu)+q_o^2(\tfrac{1}{4}+\alpha^2)\Big\}P'_{-\frac{1}{2}+i\alpha}(p_o)P'_{-\frac{1}{2}+i\alpha}(-p_o)$$

$$- 2(1-\nu)p_o^2(\tfrac{1}{4}+\alpha^2)P'_{-\frac{1}{2}+i\alpha}(p_o)\,P_{-\frac{1}{2}+i\alpha}(-p_o)$$

$$+ (\tfrac{1}{4}+\alpha^2)\Big\{1+p_o^2(1-2\nu)\Big\}P'_{-\frac{1}{2}+i\alpha}(-p_o)P_{-\frac{1}{2}+i\alpha}(p_o)$$

$$- p_o(\tfrac{1}{4}+\alpha^2)^2P_{-\frac{1}{2}+i\alpha}(p_o)\,P_{-\frac{1}{2}+i\alpha}(-p_o)\Big]\,.$$

We now superpose the two stress systems given in (117) and (119) and obtain the stress field in the cone. It is

$$
\begin{aligned}
P_{rr} &= (P_{rr})_1 + (P_{rr})_2 \,, \\
P_{\theta\theta} &= (P_{\theta\theta})_1 + (P_{\theta\theta})_2 \,, \\
P_{\varphi\varphi} &= (P_{\varphi\varphi})_1 + (P_{\varphi\varphi})_2 \,, \qquad (120) \\
P_{r\theta} &= (P_{r\theta})_1 + (P_{r\theta})_2 \,, \\
P_{r\varphi} &= 0 \quad \text{and } P_{\theta\varphi} = 0 \,.
\end{aligned}
$$

CHAPTER VII

ELASTIC CONE UNDER AN AXIAL FORCE (CONTINUED)

In this chapter the mathematics through which the stresses due to the second system of forces are obtained as discussed in Chapter VI, is described.

The equations of equilibrium are given by (91) and (92). Their solution depends upon two functions $\phi(R,Z)$ and $\Omega(R,Z)$ as stated in chapter V. Also as stated there, ϕ and Ω satisfy the equations

$$\nabla^2\Omega = \left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2} \right) = 0 \tag{121}$$

and
$$\frac{\partial^2\Omega}{\partial Z^2} = (1-\nu)\nabla^2\phi \tag{122}$$

and the corresponding stresses are given by (108). With

the help of the stresses given in (108) in cylindrical polar coordinates, we can find out the stresses $P_{ij}(i,\ j = r, \theta, \varphi)$ in spherical polar coordinates. This may be done by first transforming the right hand sides of (108) into spherical polar coordinates and then transforming the stresses in cylindrical polar coordinates to spherical polar coordinates with the help of the transformation formulae given in ((16)).

The stresses in spherical polar coordinates are

$$(P_{rr})_2 = \left(\frac{q^2}{r^2} \frac{\partial^2}{\partial p^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{2p}{r^2} \frac{\partial}{\partial p} \right)\phi$$

$$+ \frac{q^2}{r}\left(\frac{\partial}{\partial r} - \frac{p}{r} \frac{\partial}{\partial p}\right)\Omega ,$$

$$(P_{\theta\theta})_2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{p}{r^2} \frac{\partial}{\partial p}\right) \phi + \frac{p^2}{r}\left(\frac{\partial}{\partial r} - \frac{p}{r} \frac{\partial}{\partial p}\right)\Omega ,$$

$$(P_{r\theta})_2 = q\left\{\left(\frac{1}{r} \frac{\partial^2}{\partial r \partial p} - \frac{1}{r^2} \frac{\partial}{\partial p} \right) \phi + \frac{p}{r} \left(\frac{\partial}{\partial r} - \frac{p}{r} \frac{\partial}{\partial p} \right)\Omega\right\},$$

$$(P_{\varphi\varphi})_2 = \left(\frac{\partial^2}{\partial r^2} + \frac{q^2}{r^2} \frac{\partial^2}{\partial p^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{p}{r^2} \frac{\partial}{\partial p} \right) \phi$$

$$- \left\{p^2 \frac{\partial^2}{\partial r^2} + \frac{2pq^2}{r} \frac{\partial^2}{\partial r \partial p} + \frac{q^4}{r^2} \frac{\partial^2}{\partial p^2} + \frac{(1+q^2)}{r} \frac{\partial}{\partial r} \right.$$

$$\left. - \frac{p(4-3p^2)}{r^2} \frac{\partial}{\partial p} \right\}\Omega$$

$$f(r,p) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} r^{-s} \bar{f}(s,p)\, ds. \qquad (127)$$

Assuming that

$$r^{i+s} \frac{\partial^i}{\partial r^i} (\phi, \Omega) \text{ and } r^s \frac{\partial}{\partial p} (\phi, \Omega), \qquad (i = 0,1)$$

vanish as $r \to \infty$, the following relations may be easily obtained.

$$\int_0^\infty r^{s+1} \frac{\partial^2}{\partial r^2}(\phi, \Omega) dr = s(1+s)(\bar{\phi}, \bar{\Omega}),$$

$$\int_0^\infty r^s \frac{\partial}{\partial r} (\phi, \Omega)\, dr = -s(\bar{\phi}, \bar{\Omega}),$$

$$\int_0^\infty r^{s-1} \frac{\partial}{\partial p} (\phi, \Omega)\, dr = (\bar{\phi}', \bar{\Omega}'), \qquad (128)$$

$$\int_0^\infty r^{s-1} \frac{\partial^2}{\partial p^2} (\phi, \Omega)\, dr = (\overline{\phi''}, \overline{\Omega''}),$$

where dash denotes differentiation with respect to p and bar denotes the Mellin transform of the function. Multiplying equations (123), (124) and (125) by r^{s+1} and integrating with respect to r from 0 to ∞ and making use of (128), it may be seen that

$$\{\overline{r^2(P_{rr})_2}\} = q^2 \bar{\phi}'' - 2(p \bar{\phi}' + s \bar{\phi}) - q^2(p\bar{\Omega}' + s\bar{\Omega}),$$

$$\overline{\left\{r^2(P_{\theta\theta})_2\right\}} = s^2\,\bar{\phi} - p\,\bar{\phi}' - p^2(p\bar{\Omega}' + s\bar{\Omega}) ,$$

$$\overline{\left\{r^2(P_{\varphi\varphi})_2\right\}} = q^2\,\bar{\phi}'' - p\,\bar{\phi}' + s^2\,\bar{\phi} - \Big[q^4\,\bar{\Omega}'' - p\left\{2q^2 s + (4-3p^2)\right\}\bar{\Omega}' + s\left\{(s+1)\,p^2 - (2-p^2)\right\}\bar{\Omega}\Big] ,$$

$$\overline{\left\{r^2(P_{r\theta})_2\right\}} = -q\left\{(1+s)\,\bar{\phi}' + p(p\bar{\Omega}' + s\bar{\Omega})\right\}, \tag{129}$$

also
$$q^2\,\bar{\Omega}'' - 2p\,\bar{\Omega}' + s(s-1)\,\bar{\Omega} = 0, \tag{130}$$

$$q^4\bar{\Omega}'' - pq^2(3+2s)\,\bar{\Omega}' + s\left\{p^2(s+2)-1\right\}\bar{\Omega}$$
$$= (1-\nu)\left\{q^2\,\bar{\phi}'' - 2p\,\bar{\phi}' + s(s-1)\,\bar{\phi}\right\}. \tag{131}$$

Equation (130) is a Legendre's equation whose general solution is

$$\bar{\Omega} = A\,P_{s-1}(p) + B\,Q_{s-1}(p) , \tag{132}$$

where A and B are arbitrary functions of s, P_{s-1} is the Legendre function of the first kind of order s-1 and Q_{s-1} is the Legendre function of the second kind of order s-1 .

Substituting (132) in (131), we get

$$(1-\nu)\left\{q^2\,\bar{\phi}'' - 2p\,\bar{\phi}' + s(s-1)\,\bar{\phi}\right\} = s\left\{(1+2s)p^2 - s\right\}(AP_{s-1}(p) + BQ_{s-1}(p))$$
$$-pq^2(1+2s)\,(AP'_{s-1}(p) + BQ'_{s-1}(p)). \tag{133}$$

The solution of (133) is

$$\bar{\phi} = \frac{1}{(1-\nu)}\Big[C\, P_{s-1}(p) + D\, Q_{s-1}(p) - \tfrac{1}{2}\, sp^2(AP_{s-1}+BQ_{s-1})$$

$$+ \tfrac{1}{2}\, pq^2(AP'_{s-1}(p) + BQ'_{s-1}(p)\Big], \qquad (134)$$

where C and D are arbitrary constants. Legendre function of the second kind $Q_{s-1}(p)$, together with its derivatives with respect to p, have singularity at p=1. Since the stresses should remain finite (except at the point (0,0,d) on the axis), the equations (132) and (134) cannot contain $Q_{s-1}(p)$ and $Q'_{s-1}(p)$. So that B = 0 and D = 0 and the equations (132) and (134) are reduced to

$$\bar{\Omega} = A\, P_{s-1}(p) \qquad (135)$$

and

$$\bar{\phi} = \frac{1}{(1-\nu)}\Big[C\, P_{s-1}(p) - \tfrac{1}{2}\, A\, sp^2\, P_{s-1}(p) + \tfrac{1}{2}\, Apq^2\, P'_{s-1}(p)\Big]. \qquad (136)$$

On substituting (135) and (136) in (129), it may be seen that

$$\overline{\{r^2(P_{rr})_2\}} = \frac{1}{2(1-\nu)}\Big[A\,\{q^2(2\nu-2-2s)\, P'_s(p) + sp(s^2+3s+2)P_s(p)\}$$

$$- 2C(1+s)\,(q^2\, P'_s(p) + sp\, P_s(p))\Big],$$

$$\overline{\{r^2(P_{\theta\theta})_2\}} = \frac{1}{2(1-\nu)}\Big[Ap\{p(2\nu-2+s)\, P'_s(p) + s(1-s^2)P_s(p)\}$$

$$+ 2C\,\{(sq^2 - p^2)\, P'_s(p) + sp(1+s)\, P_s(p)\}\Big],$$

$$\overline{\left\{r^2(P_{\varphi\varphi})_2\right\}} = \frac{1}{2(1-\nu)}\Big[A\Big\{(2-2\nu - sp^2 - 2\nu sq^2)\,P_s'(p)$$

$$- sp(1+s)\,(1-2\nu)\,P_s(p)\Big\} + 2C\,P_s'(p)\Big],$$

$$\overline{\left\{r^2(P_{r\theta})_2\right\}} = -\frac{q}{2(1-\nu)}\Big[A\Big\{(2-2\nu - s - s^2)\,p\,P_s'(p) - s(1+s)P_s'(p)\Big\}$$

$$+ 2C(1+s)\,\{p\,P_s'(p) - s\,P_s(p)\}\Big]. \qquad (137)$$

The constants A and C are determined from the two boundary conditions given in (118) of chapter VI. We multiply (118) by r^{s+1} and integrate with respect to r from 0 to ∞. The infinite integrals on the right hand side of (118) are evaluated by using the result given below ((18))

$$\int_0^\infty \frac{r^{s-1}\,dr}{(1+2r\cos\theta+r^2)^\beta} = 2^{\beta-\frac{1}{2}}(\sin\theta)^{-\beta+\frac{1}{2}}\,\Gamma\!\left(\tfrac{1}{2}+\beta\right)B(s,2\beta-s).$$

$$\cdot\, P^{\frac{1}{2}-\beta}_{s-\beta-\frac{1}{2}}(\cos\theta) \qquad (138)$$

$$0 < \mathrm{Re}\,(s) < \mathrm{R}(2\beta),\ -\pi < \theta < \pi\ .$$

There are other results which are also used. These are given in the Appendix following this chapter.

It is sufficient to mention here that taking $-2 < \mathrm{Re}(s) < 0$ in the complex s plane, the integrals are all found to be valid and the boundary conditions in (118)

are transformed to

$$\overline{\{r^2(P_{\theta\theta})_2\}}_{\theta=\theta_0} = -H(s)\Big[\{-\mu+(\lambda+\mu)(s-1)\}q_0^2P_s'(-p_0)$$

$$-\mu p_0(1+s)P_s(-p_0)\Big], \quad (139)$$

$$\overline{\{r^2(P_{r\theta})_2\}}_{\theta=\theta_0} = -H(s)\Big[p_0q_0(\lambda+2\mu)P_s'(-p_0)+q_0(1+s)\{$$

$$-\mu+(\lambda+\mu)s\}P_s(-p_0)\Big], (140)$$

where

$$H(s) = \frac{Z_0\,\pi\,d^s}{4\pi(\lambda+2\mu)\sin\pi s}.$$

On putting the values of $\overline{\{r^2(P_{\theta\theta})_2\}}_{\theta=\theta_0}$ and $\overline{\{r^2(P_{r\theta})_2\}}_{\theta=\theta_0}$ from (139) and (140) in second and fourth equations of (137) and solving them, we get the values of the constants A and C as

$$A = \frac{H_1(s)\ \Delta_1(p_0,s)}{\Delta_3(p_0,s)} \quad (141)$$

and
$$C = -\frac{H_1(s)\ \Delta_2(p_0,s)}{\Delta_3(p_0,s)}, \quad (142)$$

where

$$H_1(s) = 2(1-\nu)\ \frac{H(s)}{s}, \quad (143)$$

$$\Delta_1(p_0,s) = 2\{(sq_0^2-p_0^2)P_s'(p_0)+s(1+s)p_0P_s(p)\}\times$$

$$\cdot\Big[(\lambda+2\mu)p_0P_s'(-p_0)+(1+s)\{-\mu+(\lambda+\mu)s\}P_s(-p_0)\Big]+$$

$$+2(1+s)(p_o P_s'(p_o) - sP_s(p_o))\Big[\{-\mu + (\lambda+\mu)(s-1)\}\, q_o^2\, P_s'(-p_o)$$

$$-\mu p_o(1+s)\, P_s(-p_o)\Big], \qquad (144)$$

$$\Delta_2(p_o,s) = \Big[(\lambda+2\mu)p_o P_s'(-p_o) + \{-\mu+(\lambda+\mu)s\}\,(1+s)\,P_s(-p_o)\Big]\cdot$$

$$\cdot\left\{p_o^2(s-2+2\nu)\,P_s'(p_o) + sp_o(1-s^2)P_s(p_o)\right\}$$

$$+\Big[\{-\mu + (\lambda+\mu)(s-1)\}\,q_o^2\,P_s'(-p_o) - \mu p_o(1+s)P_s(-p_o)\Big]\cdot$$

$$\cdot\Big[\{2(1-\nu) - s(1+s)\}\,p_o\,P_s'(p_o) - s(1+s)\,P_s(p_o)\Big],$$

$$(145)$$

$$\Delta_3(p_o,s) = \{2(1-\nu) - q_o^2 s(1+s)\}\,p_o\,P_s'(p_o)\,P_s'(p_o)$$

$$-s(1+s)\,q_o^2\,P_s'(p_o)\,P_s(p_o) - s^2(1+s)^2 p_o P_s(p_o)P_s(p_o)\,.$$

$$(146)$$

Substituting the values of A and C from (141) and (142) in (137) and inverting with the help of the inversion formula (127), the stresses in complex integral form are given by

$$(P_{rr})_2 = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{H(s)\,r^{-s-2}}{s\,\Delta_3(p_o,s)}\Big[\Delta_1(p_o,s)\{q^2(2\nu-2-2s)P_s'(p)$$

$$+p(1+s)(2+s)s\,P_s(p)\} + 2\Delta_2(p_o,s)(1+s)\{q^2P_s'(p)+spP_s'(p)\}\Big]ds,$$

$$(P_{\theta\theta})_2 = \frac{1}{2\pi\, i}\int_{\surd - i\infty}^{\surd + i\infty} \frac{H(s)\, r^{-s-2}}{s\,\Delta_3(p_o,s)}\Big[\Delta_1(p_o,s)\{p^2(s-2+2\nu)P_s'(p)$$

$$+ps(1-s^2)\,P_s(p)\} - 2\Delta_2(p_o,s)\{s(1+s)p\,P_s(p)$$

$$+ (sq^2 - p^2)\,P_s'(p)\}\Big]\,ds \quad ,$$

$$(P_{\varphi\varphi})_2 = \frac{1}{2\pi i}\int_{\surd - i\infty}^{\surd + i\infty} \frac{H(s)\, r^{-s-2}}{s\,\Delta_3(p_o,s)}\Big[\Delta_1(p_o,s)\{(2-2\nu - sp^2 - 2\nu sq^2)P_s'(p)$$

$$- s(1+s)\,(1-2\nu)\,p\,P_s(p)\} - 2\,\Delta_2(p_o,s)\,P_s'(p)\Big]\,ds \; ,$$

$$(P_{r\theta})_2 = -\frac{q}{2\pi i}\int_{\surd - i\infty}^{\surd + i\infty} \frac{H(s) r^{s-2}}{s\,\Delta_3(p_o,s)}\Big[\Delta_1(p_o,s)\cdot$$

$$\cdot\{(2-2\nu - s - s^2)p\,P_s'(p) - s(1+s)\,P_s(p)\}$$

$$- 2\Delta_2(p_o,s)\,(1+s)(p\,P_s'(p) - s\,P_s(p)\,)\Big]\,ds \; ,$$

(147)

$(P_{\theta\varphi})_2 = 0$ and $(P_{r\varphi})_2 = 0$.

The solution obtained in complex integral form (147) is not affected by the particular choice of $\surd$, as long as $Re(s)$ ($= \surd$) remains within the limits $-2 < Re(s) < 0$. The choice of $\surd = -1/2$, which in turn implies

$$s = -\frac{1}{2} + i\alpha \; , \qquad -\infty < \alpha < \infty \qquad (148)$$

is suggested by the fact that the Legendre function $P_s(p)$ together with its derivative is real valued for $s = -\frac{1}{2} + i\alpha$, as may be inferred from the result

$$P_{-\frac{1}{2}+i\alpha}(\cos\theta) = \frac{2}{\pi} \cosh \pi\alpha \int_0^\infty \frac{\cos \alpha x \, dx}{(2\cosh x + 2\cos\theta)^{1/2}},$$

$$-\pi < \theta < \pi . \tag{149}$$

In order to establish the validity of our choice of $\sqrt{\ }$, we have yet to show that the integrands in (147), regarded as functions of s, are free from the singularities if (148) holds. The singularities of the integrand may arise from the poles of $H_1(s)$ and zeros of $\Delta_3(p_o,s)$. The poles of $H_1(s)$ are, in fact, the zeros of $s \sin \pi s$. It may be noted that $s \sin s\pi$ does not vanish for s given by (148).

It may be seen that $\Delta_3(p_o, s)$ is also free from zeros by considering the function $\Delta_3(p_o, \alpha)$, instead of $\Delta_3(p_o, s)$. Differentiating $\Delta_3(p_o, \alpha)$ with respect to θ_o

$$\frac{\partial \Delta_3(p_o, \alpha)}{\partial \theta_o} = -\frac{1}{q_o} P^1_{-\frac{1}{2}+i\alpha}(p_o) \Big[(1+\nu)(\tfrac{1}{4} + \alpha^2) P^1_{-\frac{1}{2}+i\alpha}(p_o) + (1-\nu) P^3_{-\frac{1}{2}+i\alpha}(p_o) \Big], \quad 0 < \theta_o < \pi . \tag{150}$$

Using the result

$$P_\beta^{\,m}(\cos\theta) = (-1)^m (1-\cos^2\theta)^{m/2} \frac{d^m}{d(\cos\theta)^m} \{P_\beta(\cos\theta)\} \qquad (151)$$

and in (149) , it may be concluded that

$$P^m_{-\frac{1}{2}+i\alpha}(p_o) > 0 \qquad (\text{for } m = 1, 3 \quad -\infty < \alpha < \infty). \qquad (151)$$

Consequently, (150) yields

$$\frac{\partial \, \Delta_3(p_o,\alpha)}{\partial\theta_o} < 0 \qquad 0 < \theta_o < \pi \quad .$$

Also as $\theta_o \to 0$

$$\Delta_3(p_o, \alpha) = -\frac{1}{2}(1+\nu)(\frac{1}{4}+\alpha^2)^2 \text{ , i.e. } < 0 \text{ .}$$

Hence, it may be concluded that $\Delta_3(p_o, \alpha) < 0$ for $0 < \theta_o < \pi$ and $-\infty < \alpha < \infty$. We now put $s = -\frac{1}{2} + i\alpha$ in (147) and separate the integrals in real and imaginary parts. The real and imaginary parts are found to be even and odd functions of α, respectively. Consequently, the contribution of the imaginary part is zero and the stresses $(P_{rr})_2$, $(P_{\theta\theta})_2$, $(P_{\varphi\varphi})_2$,$(P_{r\theta})_2$, $(P_{\theta\varphi})_2$ and $(P_{r\varphi})_2$ are obtained as given by equation (119).

It may be noted that while solving this problem of an axial force in the cone, another auxiliary problem when the tractions are prescribed on the surface of the cone is also solved. Hence all those problems in which tractions are prescribed on the surface of the cone can be tackled with the help of the method given for this problem after suitable modifications.

In the next chapter, the convergence of the integrals in (119) is discussed. To have a feeling for the stress distribution, graphs and tables are given for some particular cases.

APPENDIX TO CHAPTER VII

The following results for our use can be deduced as particular cases of the result given in (138).

$$\int_0^\infty \frac{r^{s+3}dr}{(r^2+d^2-2drp)^{5/2}} = \frac{\pi\, d^{s-1}}{3q^2\sin\pi s}\Big\{-\ (2+sq^2)P_s'(-p) + ps(1+s)P_s(-p)\Big\},$$
$$-4 < Re(s) < 1$$

$$\int_0^\infty \frac{r^{s+2}}{(r^2+d^2-2drp)^{5/2}} = -\frac{\pi d^{s-2}}{3q^2\sin\pi s}\Big\{2p\ P_s'(-p)-s(1+s)P_s(-p)\Big\},$$
$$-3 < Re(s) < 2$$

$$\int_0^\infty \frac{r^{s+1}\,dr}{(r^2+d^2-2drp)^{5/2}} = \frac{\pi\, d^{s-3}}{3q^2\sin\pi s}\Big\{(q^2(s-1)-2p^2)P_s'(-p) + p\ s(1+s)\ P_s(-p)\Big\},$$
$$-2 < Re(s) < 3$$

$$\int_0^\infty \frac{r^{s+2}\,dr}{(r^2+d^2-2drp)^{3/2}} = \frac{\pi\, d^s}{\sin\pi s}\Big\{-\ p\ P_s'(-p)-(1+s)P_s(-p)\Big\},$$
$$-3 < Re(s) < 0$$

$$\int_0^\infty \frac{r^{s+1}}{(r^2+d^2-2drp)^{3/2}} = -\frac{\pi\, d^{s-1}}{\sin\pi s}\ P_s'(-p),$$
$$-2 < Re(s) < 1$$

$$-1 < p < 1.$$

The relations given below have been used to derive the above results.

$$P'_{s+1}(p) - P'_{s-1} = (2s+1)\, P_s(p) ,$$

$$p\, P'_s(p) - P'_{s-1}(p) = s\, P_s(p) ,$$

$$(1-p^2)\, P'_s(p) = (s+1)\, p\, P_s(p) - (s+1)\, P_{s+1}(p),$$

$$(s+1)\, P_{s+1}(p) = + (2s+1)p\, P_s(p) - s\, P_{s-1}(p) ,$$

$$P_s^{-m}(p) = (-1)^m \frac{\Gamma(s-m+1)}{\Gamma(s+m+1)} P_s^m (p),$$

$$P_s^m(p) = (-1)^m (1-p^2)^{m/2} \frac{d^m}{d(p)^m} P_s(p),$$

$$B(s,\, 1-s) = \frac{\pi}{\sin \pi s} ,$$

where B denotes the Beta function, m is a positive integer, s may be complex and $p = \cos\theta$ $(-\pi < \theta < \pi)$.

CHAPTER VIII

NUMERICAL EVALUATION AND DISCUSSION

The convergence of the integrals given in (119), will now be discussed. The asymptotic expansion of $P_{-\frac{1}{2}+i\alpha}(\cos\theta)$ and $P'_{-\frac{1}{2}+i\alpha}(\cos\theta)$ may be derived from the following results ((19));

$$P_s^m(\cos\theta) = \sqrt{\frac{2}{\pi q}}\ \frac{\Gamma(s+m+1)}{\Gamma(s+\frac{3}{2})} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}(\frac{1}{2}+m)_{\ell}(\frac{1}{2}-m)_{\ell}}{\underline{\ell}\,(2\sin\theta)^{\ell}\,(s+\frac{3}{2})_{\ell}} \cdot$$

$$\cdot \sin\left\{(s+\ell+\tfrac{1}{2})\theta + (\tfrac{m}{2}+\tfrac{1}{4})\pi + \tfrac{1}{2}\ell\pi\right\} \qquad (152)$$

$$\text{as } |s| \to \infty \quad \epsilon \leq \theta \leq \pi-\epsilon,\ \epsilon > 0$$

and

$$\frac{\Gamma(s+m)}{\Gamma(s+n)} = s^{m-n}\left[1 + \frac{(m-n)(m+n-1)}{2s} + O(s^{-2})\right] \qquad (153)$$

as $|s| \to \infty$, m, n real, $-\pi < \arg(s) < \pi$.

Using (152) and (153), following results may be obtained .

$$P_{-\frac{1}{2}+i\alpha}(p) = \frac{\exp(\alpha.\theta)}{\sqrt{2\pi\alpha\, q}}\left\{1 + \frac{p}{8\alpha q} + O(\alpha^{-2})\right\}, \qquad (154)$$

$$\epsilon \leqslant \theta \leqslant \pi - \epsilon\ ,\quad \epsilon > 0,\quad \alpha \to \infty$$

and

$$P'_{-\frac{1}{2}+i\alpha}(p) = -\sqrt{\frac{\alpha}{2\pi}}\ \frac{\exp(\alpha.\theta)}{q^{3/2}}\left\{1 - \frac{3p}{8\alpha q} + O(\alpha^{-2})\right\}, \qquad (155)$$

$$\epsilon \leqslant \theta \leqslant \pi - \epsilon\ ,\quad \epsilon > 0,\quad \alpha \to \infty\ .$$

On the axis of the cone, i.e. for $\theta = 0$

$$P_{-\frac{1}{2}+i\alpha}(1) = 1 \qquad \text{and} \qquad P'_{-\frac{1}{2}+i\alpha}(1) = -\frac{1}{2}\left(\frac{1}{4} + \alpha^2\right).$$

If each of the integrands in (119) is denoted by $f(\alpha, p, p_o)$, then making use of the results (154) and (155), it may be seen that

$$f(\alpha, p, p_o) = O\left(\alpha^{1/2} \exp\{-\alpha(2\theta_o - \theta)\}\right) \qquad (156)$$

as $\alpha \to \infty$.

This completes the proof for convergence.

It seems necessary to find the stress distribution, for some particular cases. This has been done by evaluating the integrals in (119) numerically for some cases. The method of numerical solution of these integrals may be divided into two parts. The first is the calculation of the Legendre functions of the type $P_{-\frac{1}{2}+i\alpha}(p)$ and their derivatives. The series expansion $P_{-\frac{1}{2}+i\alpha}(p)$ given in ((20)) and in (157) below has been employed for calculation.

$$P_{-\frac{1}{2}+i\alpha}(p) = F\left(\tfrac{1}{2} - i\alpha,\ \tfrac{1}{2} + i\alpha\ ;\ 1 : \tfrac{1-p}{2}\right), \qquad (157)$$

$$-1 < p < 1 .$$

$P'_{-\frac{1}{2}+i\alpha}(p)$ may be obtained directly. The series on the right hand side of (157) can be calculated upto any degree of accuracy.

The second part is the evaluation of the infinite integrals. The presence of oscillatory functions $\sin m\alpha$ and $\cos m\alpha$ suggests the use of Filon's method ((21)). However, if the increment is small, one may employ even the well known Simpson's rule. Filon's formula is given below :

Let $I_1 = \int_a^b f(\alpha) \sin m'\alpha \, d\alpha$.

If the interval (a, b) is divided into 2n intervals each of length h then, approximately,

$$I_1 = h\left[a_1(f(a) \cos(m'a) - f(b) \cos(m'b)) + a_2 S_{2r} + a_3 S_{2r-1} \right] ,$$

where

$$S_{2r} = \sum_{r=0}^{\infty} \left[f(a+2rh) \sin \{ m'(a+2rh) \} - \tfrac{1}{2} \{ f(a) \sin m'a + f(b) \sin m'b \} \right] ,$$

$$S_{2r-1} = \sum_{r=0}^{\infty} f\{ a + (2r-1)h \} \sin m'\{ a + (2r-1)h \} ,$$

$$a_1 = \frac{1}{\beta_1} + \frac{\sin 2\beta_1}{2\beta_1^2} - \frac{2\sin^2\beta_1}{\beta_1^3} ,$$

$$a_2 = 2\left\{ \frac{1+\cos^2\beta_1}{\beta_1^2} - \frac{\sin 2\beta_1}{\beta_1^3} \right\} , \qquad (158)$$

$$a_3 = 4\left(\frac{\sin\beta_1}{\beta_1^3} - \frac{\cos\beta_1}{\beta_1^2} \right) ,$$

where $\beta_1 = m'h$.

Simpson's rule is rather well known and is not given here.

The upper limit of the infinite integral may be taken as some finite value of α which is not arbitrary and is to be chosen suitably bearing in mind the result in (156). It may be remarked that value of α may be suitably adjusted depending upon the semi-vertex angle of the cone.

The interval length is taken to be 0.1. Numerical work has been done for two values of the semi-vertex angles $\theta_o = 30^o$ and 45^o. The values assigned to θ are 0^o and θ_o. The value of ν is taken as 0.25. For each of the above cases the stresses P_{rr}, $P_{\theta\theta}$, $P_{\varphi\varphi}$ and $P_{r\theta}$ have been calculated. It may be noted that the resultant stresses are the sum of two systems of stresses $(P_{rr})_1$ etc. and $(P_{rr})_2$ etc. The first system is not complicated and may be calculated to a fair degree of accuracy. The second system of stresses involve infinite integrals and complicated integrands. Both Filon's method and Simpson's rule, were employed and the results obtained by both the methods agree closely. It may be remarked here, that the boundary conditions are satisfied.

The results of our problem for $d = 0.01$ were calculated and compared with the well known results for $d = 0$ given in ((16)). It was felt that the two results cannot differ significantly. The values assigned

to r for $d = 0.01$ are 0.1, 1.0, 2.0, 3.0, and 4.0. It is interesting to note from Table I, p.134 that the results agree closely with maximum error of 1 percent. This provides a further check on the numerical solution of the problem.

Graphs of P_{rr}, $P_{\theta\theta}$, $P_{\varphi\varphi}$ and $P_{r\theta}$ on the surface as well as on the axis of the cones for $\theta_0 = 30^\circ$ and 45° have been drawn. The values given to d are 1.0 and 2.0 and for each value of d, r takes the values 0.5, 1.5, 2.5, 3.5, 4.5, 5.5, 6.5 and 7.5 . Four graphs are given on p.135, 136,137,138 . The graphs are self explanatory.

TABLE 1

$\theta_o = 30^o$; $\nu = 0.25$.

θ	r	$P_{rr} \cdot \pi/Z_o$		$P_{\theta\theta} \cdot \pi/Z_o$		$P_{\phi\phi} \cdot \pi/Z_o$		$P_{r\theta} \cdot \pi/Z_o$	
		In present problem	In solution given in Love	In present problem	In solution given in Love	In present problem	In solution given in Love	In present problem	In solution given in Love
θ_o	r= .1	-358.69	-358.68	0.0020	0.0000	-11.460	-11.452	0.0021	0.0000
	r= 1	-3.5868	-3.5868	0.0000	0.0000	- 0.1145	-0.1145	0.0000	0.0000
	r= 2	-0.8967	-0.8967	0.0000	0.0000	-0.0286	-0.0286	0.0000	0.0000
	r= 3	-0.3985	-0.3985	0.0000	0.0000	-0.0127	-0.0127	0.0000	0.0000
	r= 4	-0.2241	-0.2241	0.0000	0.0000	-0.0071	-0.0071	0.0000	0.0000
15^o	r= .1	-418.43	-418.46	4.1897	4.1957	1.4202	1.4310	1.1354	1.1242
	r= 1	-4.1846	-4.1846	0.0419	0.0419	0.0143	0.0143	0.0112	0.0112
	r= 2	-1.0461	-1.0461	0.0104	0.0104	0.0035	0.0035	0.0028	0.0028
	r=3	-0.4649	-0.4649	0.0046	0.0046	0.0015	0.0015	0.0012	0.0012
	r= 4	-0.2615	-0.2615	0.0026	0.0026	0.0008	0.0008	0.0007	0.0007
	r= .1	-438.82	-438.84	5.7121	5.7260	5.7121	5.7260	0.0000	0.0000
	r= 1	-4.3884	-4.3884	0.0572	0.0572	0.0572	0.0572	0.0000	0.0000
	r= 2	-1.0971	-1.0971	0.0143	0.0143	0.0143	0.0143	0.0000	0.0000
	r= 3	-0.4876	-0.4876	0.0063	0.0063	0.0063	0.0063	0.0000	0.0000
	r= 4	-0.2742	-0.2742	0.0035	0.0035	0.0035	0.0035	0.0000	0.0000

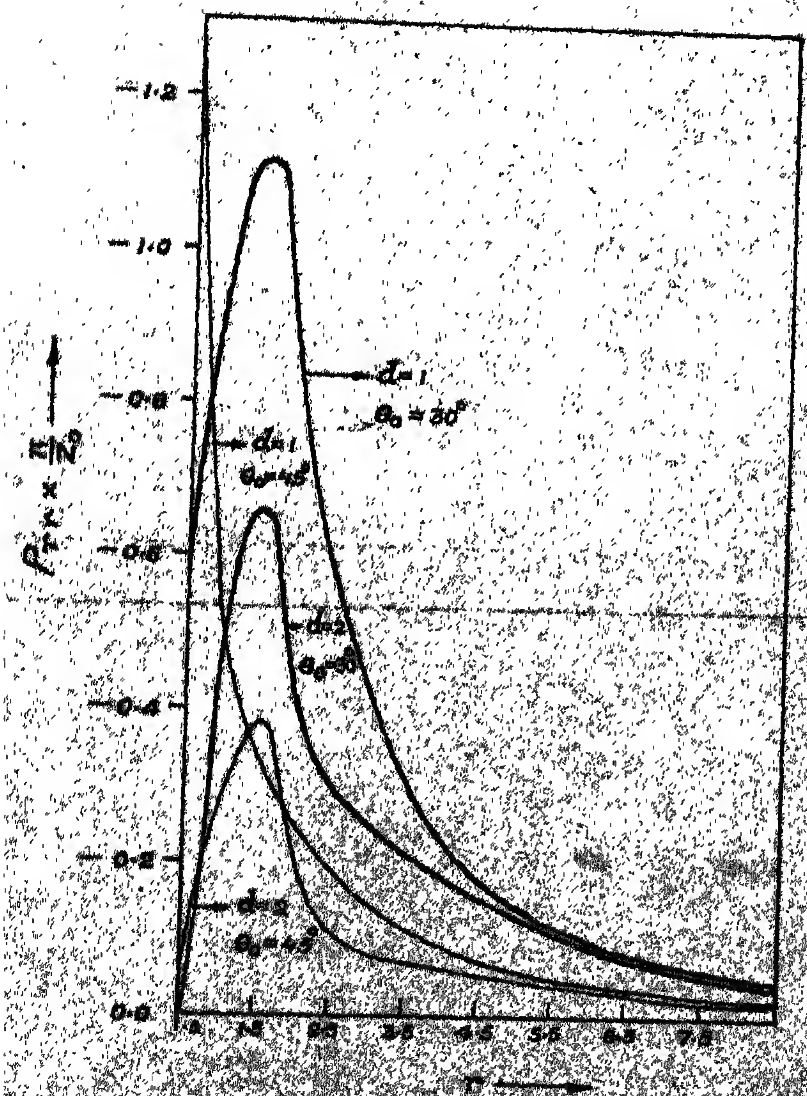


FIG-1- P_{rr} ALONG THE SURFACES OF THE CONES FOR $\theta_0=30°$ AND $\theta_0=45°$ AND $\nu=\cdot 25$

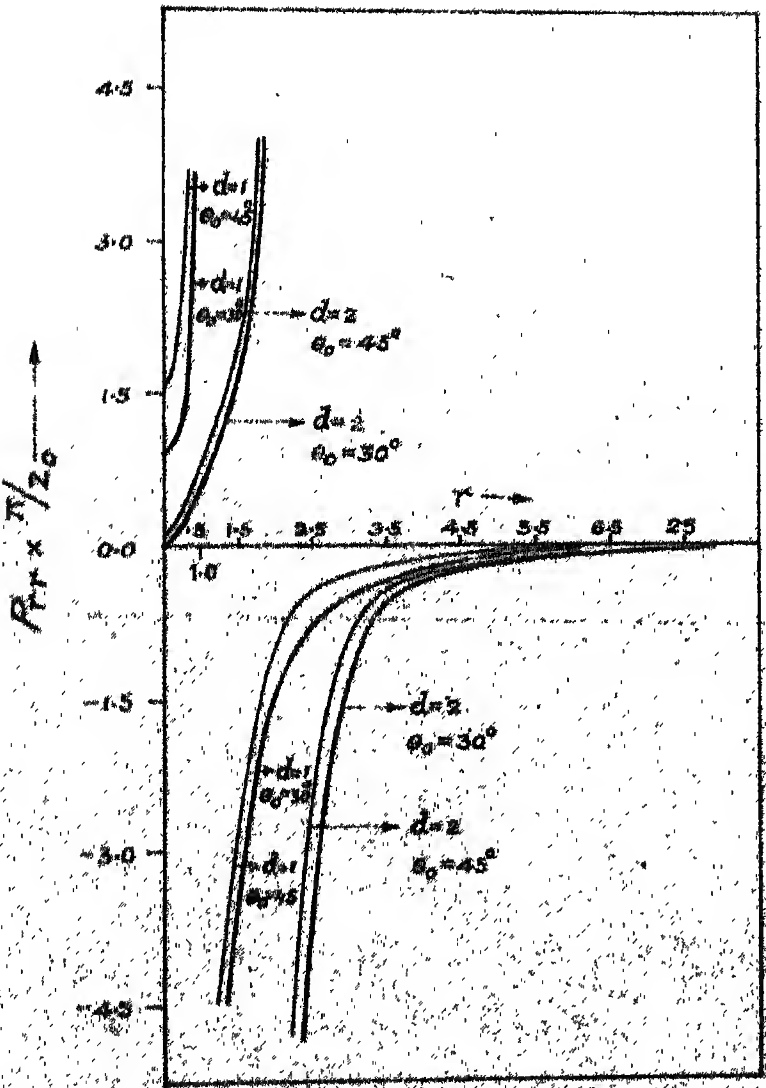


FIG. 2 – P_{rr} ALONG THE AXES OF THE CONES FOR $\theta_0=30°$ AND $\theta_0=45°$ AND $\nu=\cdot25$

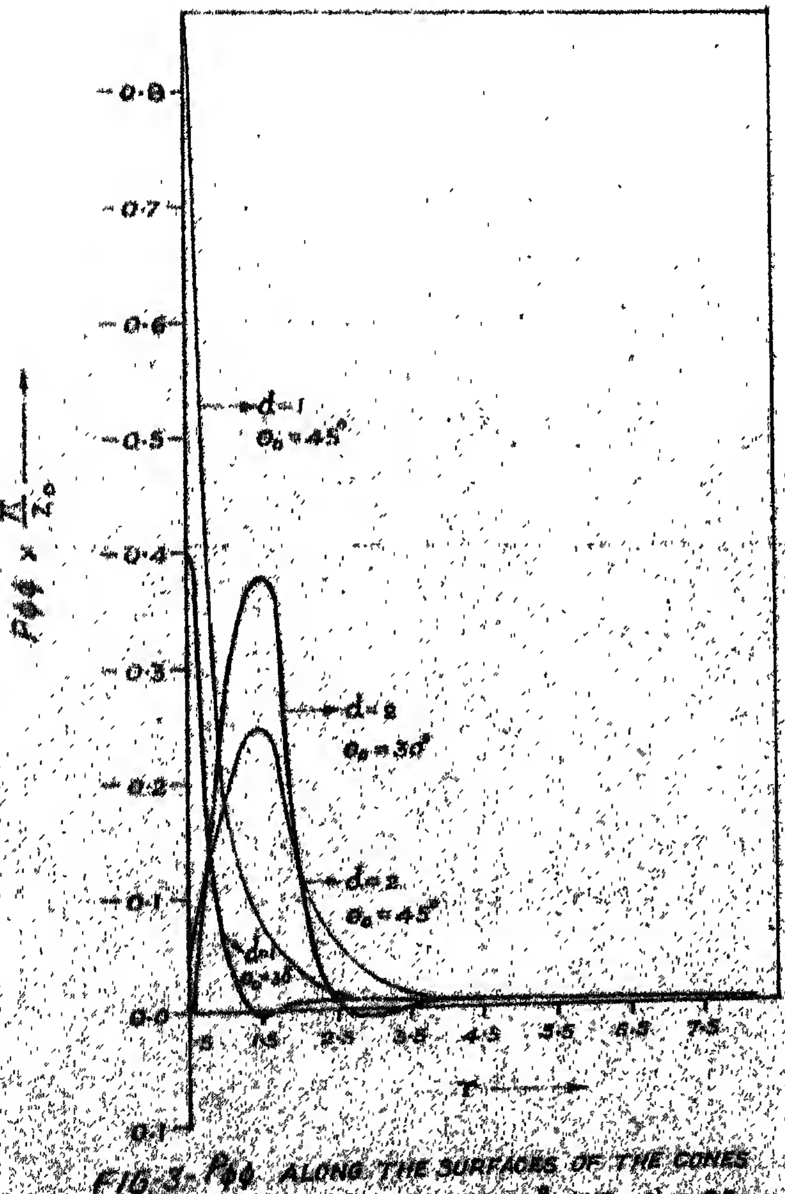


FIG. 3 - $P_{\phi\phi}$ ALONG THE SURFACES OF THE CONES FOR $\theta_0 = 30°$ AND $\theta_0 = 45°$ AND $\nu = .25$

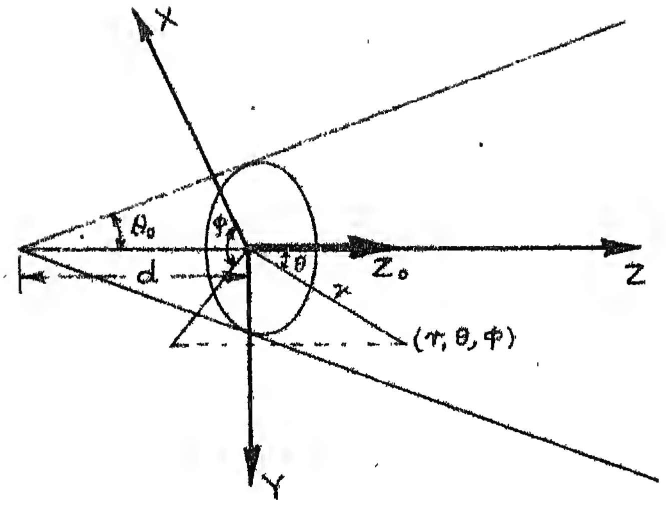


FIG-1- AXIAL FORCE Z_0 IN A CONE AND THE COORDINATE SYSTEM (r, θ, ϕ).

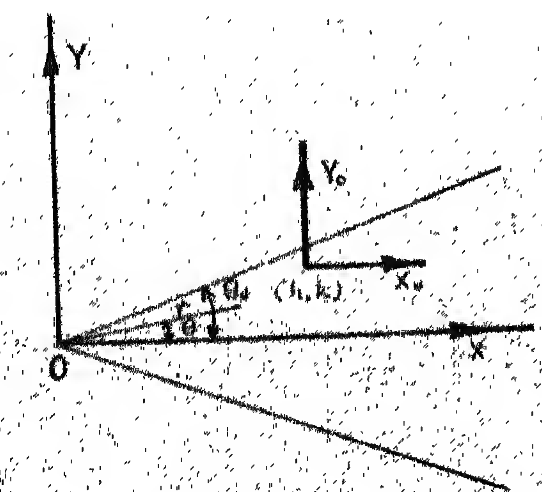


FIG.-2- POINT FORCE (X_0, Y_0) IN A WEDGE AND COORDINATE SYSTEMS (X,Y) AND (r, θ).

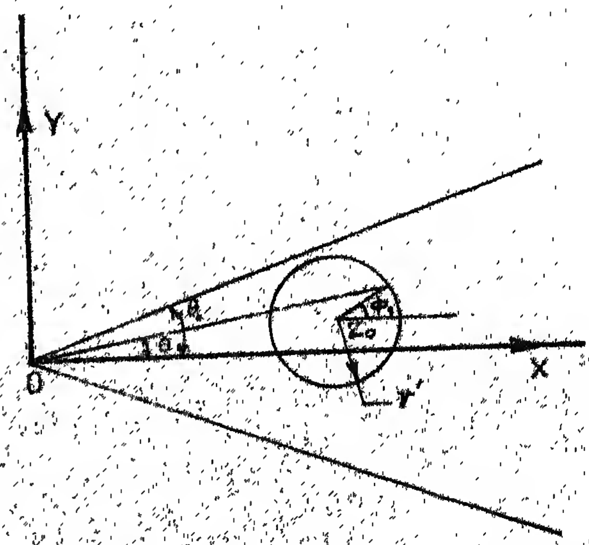


FIG.-3- CIRCULAR INCLUSION IN WEDGE AND COORDINATE SYSTEMS (X,Y) AND (r, θ).

CHAPTER IX

ELASTIC CYLINDER UNDER AN AXIAL FORCE

The problem of determining stresses due to a point force along the axis in an infinite elastic cylinder is considered in this chapter. The problem may be stated as follows :

Let a point force $(0, 0, Z_0)$ act at the origin of coordinates in the direction of the axis of an infinite cylinder. The radius of the cylinder is a and its axis is taken along the z axis $(-\infty < z < \infty)$ (Figure 1 ,p.155). The solution to the above problem has been achieved by the same process which has been described for the cone in chapter 6, p.102 . The first system of stresses are obtained by applying a point force in an infinite elastic medium. The solution of this is known from ((16)). Note that in this case the point force

acts at (0,0,0). The displacement components (u_x, u_y, u_z) in Cartesian coordinates when a point force Z_o acts at (0,0,0) may be derived from (113) by putting x_1, y_1, z_1 to be zero and $(X_o, Y_o, Z_o) = (0,0, Z_o)$. They are

$$u_x = \frac{(\lambda+\mu)\ Z_o}{8\pi\mu(\lambda+2\mu)} \cdot \frac{xz}{r_3^3},$$

$$u_y = \frac{(\lambda+\mu)\ Z_o}{8\pi\mu(\lambda+2\mu)} \cdot \frac{yz}{r_3^3}, \tag{159}$$

$$u_z = \frac{(\lambda+\mu)\ Z_o}{8\pi\mu(\lambda+2\mu)} \cdot \frac{z^2}{r_3^3} + \frac{(\lambda+3\mu)\ Z_o}{8\pi\mu(\lambda+2\mu)} \cdot \frac{1}{r_3}$$

where $r_3^2 = x^2 + y^2 + z^2$.

The displacement components in cylindrical polar coordinates (R, θ, Z) $(x = R\cos\theta,\ y = R\sin\theta,\ z = Z,\ 0 \leq \theta < 2\pi,\ -\infty < Z < \infty)$ are

$$u_R = \frac{(\lambda+\mu)\ Z_o}{8\pi\mu(\lambda+2\mu)} \frac{RZ}{(R^2+Z^2)^{3/2}},$$

$$u_Z = \frac{(\lambda+3\mu)Z_o}{8\pi\mu(\lambda+2\mu)} \frac{1}{(R^2+Z^2)^{1/2}} + \frac{(\lambda+\mu)\ Z_o}{8\pi\mu(\lambda+2\mu)} \frac{Z^2}{(R^2+Z^2)^{3/2}},$$

$$u_\theta = 0. \tag{160}$$

Stresses corresponding to (160) in cylindrical polar coordinates are

$$(P_{RR})_1 = \frac{\mu Z_o}{4\pi(\lambda+2\mu)} \frac{Z}{(R^2+Z^2)^{3/2}} - \frac{3(\lambda+\mu) Z_o}{4\pi(\lambda+2\mu)} \frac{R^2 Z}{(R^2+Z^2)^{5/2}},$$

$$(P_{\theta\theta})_1 = \frac{\mu Z_o}{4\pi(\lambda+2\mu)} \frac{Z}{(R^2+Z^2)^{3/2}},$$

$$(P_{ZZ})_1 = \frac{3(\lambda+\mu) Z_o}{4\pi(\lambda+2\mu)} \frac{R^2 Z}{(R^2+Z^2)^{5/2}} - \frac{(3\lambda+4\mu) Z_o}{4\pi(\lambda+2\mu)} \frac{Z}{(R^2+Z^2)^{3/2}},$$

$$(P_{RZ})_1 = \frac{3(\lambda+\mu) Z_o}{4\pi(\lambda+2\mu)} \frac{R^3}{(R^2+Z^2)^{5/2}} - \frac{(3\lambda+4\mu) Z_o}{4\pi(\lambda+2\mu)} \frac{R}{(R^2+Z^2)^{3/2}},$$

$$(P_{R\theta})_1 = 0 \quad \text{and} \quad (P_{\theta Z})_1 = 0. \tag{161}$$

The subscript one is used to the above stresses in (161) to indicate that they are the first system of stresses.

Now if the point force Z_o acts at (0,0,0) in the cylinder, then on the surface of the cylinder, the normal and shearing stresses are zero i.e. $P_{RR} = 0$, $P_{RZ} = 0$ and $P_{R\theta} = 0$ at $R = a$. So a second system of stresses $(P_{RR})_2$, $(P_{RZ})_2$, $(P_{\theta\theta})_2$, $(P_{ZZ})_2$, $(P_{R\theta})_2$ and $(P_{\theta Z})_2$ is

superposed on the first stress system such that on the surface of the cylinder

$$(P_{RR})_2 = -(P_{RR})_1,\ (P_{RZ})_2 = -(P_{RZ})_1 \text{ and } (P_{R\theta})_2 = -(P_{R\theta})_1 = 0.$$

Putting R = a in $(P_{RR})_1$, $(P_{RZ})_1$ in (161), we get

$$\left\{(P_{RR})_2\right\}_{R=a} = -\frac{\mu Z_o}{4\pi(\lambda+2\mu)} \frac{Z}{(a^2+Z^2)^{3/2}} + \frac{3(\lambda+\mu)Z_o}{4\pi(\lambda+2\mu)} \frac{a^2 Z}{(a^2+Z^2)^{5/2}}, \tag{162}$$

$$\left\{(P_{RZ})_2\right\}_{R=a} = -\frac{3(\lambda+\mu)Z_o}{4\pi(\lambda+2\mu)} \frac{a^3}{(a^2+Z^2)^{5/2}} + \frac{(3\lambda+4\mu)Z_o}{4\pi(\lambda+2\mu)} \frac{a}{(a^2+Z^2)^{3/2}}. \tag{163}$$

The problem now is to find the stresses within the cylinder with the two boundary conditions as given in (162) and (163). To solve this, we start with the equations of equilibrium in cylindrical coordinates (R, θ, Z). For axisymmetric case (variation with respect to θ is zero) in the absence of body forces; they are given in (91) and (92). The solution of these equilibrium equations in terms of two functions $\phi(R,Z)$ and $\Omega(R,Z)$ is given in chapter V . The stresses in terms of ϕ and Ω

are given by (108) . ϕ and Ω satisfy (112).

The exponential Fourier transform of a function $f(R,Z)$, denoted by $\bar{f}\,(R,\alpha)$ is defined as ((17))

$$\bar{f}(R,\alpha) = \int_{-\infty}^{\infty} f(R,Z)\, e^{i\alpha Z}\, dZ \qquad (164)$$

and the inverse Fourier transform is given by

$$f(R,Z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}\,(R,\alpha)\, e^{-i\alpha Z}\, d\alpha\ . \qquad (165)$$

Assuming that

$$e^{i\alpha Z} \quad \frac{\partial}{\partial Z} \ (\phi,\Omega) \text{ and } \ e^{i\alpha Z}\ (\phi,\Omega) \text{ vanish } \ \text{as } z \to \pm\infty$$

and taking the Fourier transform as defined in (164) of the equations in (112), it may be seen that the solutions of the transformed equations which are finite at $R = 0$, are

$$\bar{\Omega} = A(\alpha)\ I_o(R\alpha) \qquad (166)$$

and

$$\bar{\phi} = \frac{1}{2(1-\nu)} \left\{ 2\ B(\alpha)\ I_o(R\alpha) - A(\alpha)\ R\ I_o'(R\alpha) \right\} \ , \qquad (167)$$

bar denotes the exponential Fourier transform and dash stands for the differentiation with respect to R and not with respect to the argument ; $A(\alpha)$ and $B(\alpha)$ are the

constants to be determined from the boundary conditions, I_o is the modified Bessel function of the first kind and order zero.

Taking the Fourier transform of $(P_{RR})_2$ etc. in (108) and substituting the values of $\bar{\phi}$ and $\bar{\Omega}$ from (166) and (167), it may be seen that

$$2R(1-\nu)(\overline{P_{RR}})_2 = A(\alpha)\left[\{2(1-\nu)+R^2\alpha^2\} I_o'(R\alpha) - R\alpha^2 I_o(R\alpha)\right]$$

$$+ 2B(\alpha)\left[I_o'(R\alpha) - R\alpha^2 I_o(R\alpha)\right] ,$$

$$2R(1-\nu)(\overline{P_{\theta\theta}})_2 = A(\alpha)\left\{(1-2\nu)R\alpha^2 I_o(R\alpha) - 2(1-\nu) I_o'(R\alpha)\right\}$$

$$- 2B(\alpha) I_o'(R\alpha) ,$$

$$2R(1-\nu)(\overline{P_{ZZ}})_2 = -A(\alpha)\left\{2R\alpha^2 I_o(R\alpha) + R^2\alpha^2 I_o'(R\alpha)\right\}$$

$$+ 2R\alpha^2 B(\alpha) I_o(R\alpha) ,$$

$$2R(1-\nu)(\overline{P_{RZ}})_2 = -iR^2\alpha^3 A(\alpha) I_o(R\alpha) + 2iR\alpha B(\alpha) I_o'(R\alpha) , \tag{168}$$

$$(\overline{P_{R\theta}})_2 = 0 \quad \text{and} \quad (\overline{P_{\theta Z}})_2 = 0 .$$

Taking the Fourier transform of (162) and (163), we get

$$\left\{\overline{(P_{RR})_2}\right\}_{R=a} = -\frac{i\ Z_o}{2\pi\ (\lambda+2\mu)}\Big\{\mu\alpha\ K_o(a|\alpha|)$$

$$+ (\lambda+\mu)\, a\alpha\ K_o'(a|\alpha|)\Big\},$$

(169)

$$\left\{\overline{(P_{RZ})_2}\right\}_{R=a} = -\frac{Z_o}{2\pi\ (\lambda+2\mu)}\Big\{ (\lambda+2\mu)\, K_o'\ a|\alpha|$$

$$+ (\lambda+\mu)\, a\ \alpha^2 K_o(a|\alpha|)\Big\} \quad ,(170)$$

K_o is the modified Bessel function of the second kind and order zero .

The following result has been used in (169) and (170)

$$\int_{-\infty}^{\infty} \frac{e^{ifg}\, df}{(\alpha^2+f^2)^{\beta}} = \frac{2\pi^{1/2}|g|^{\beta-1/2}}{\Gamma(\beta)\ (2\alpha)^{\beta-1/2}}\ K_{\beta-1/2}(\alpha|g|),$$

(171)

given in ((22)) and recurrence relations in ((23)).

From first and fourth equations of (168) and the boundary conditions (169) and (170), the values of the constants A and B may be determined. They are

$$A(\alpha) = \frac{2i\ Z_o(1-\nu)a}{\pi\ (\lambda+2\mu)\,\Delta(\alpha)}\Big[\{(\lambda+2\mu) + a^2\alpha^2(\lambda+\mu)\}\ K_o'(a|\alpha|)I_o'(a\alpha)$$

$$- a^2\alpha^4(\lambda+\mu)K_o(a|\alpha|)\ I_o(a\alpha) + (\lambda+2\mu)\ \alpha^2\Big],(172)$$

and

$$B(\alpha) = -\frac{iZ_0(1-\nu)\ a}{\pi(\lambda+2\mu)\Delta(\alpha)}\Big[\ (\lambda+2\mu)\left\{2(1-\nu)+a^2\alpha^2\right\}K_0'(a|\alpha|)I_0'(a\alpha)$$

$$-(\lambda+2\mu)\ a^2\alpha^4\ I_0(a\alpha)\ K_0(a|\alpha|) + \alpha^2\left\{(\lambda+2\mu)+a^2\alpha^2(\lambda+\mu)\right\}\Big], \tag{173}$$

where $$\Delta(\alpha) = -2a\alpha\left[\left\{2(1-\nu)+a^2\alpha^2\right\}\left\{I_0'(a\alpha)\right\}^2 - a^2\alpha^4\left\{I_0(a\alpha)\right\}^2\right]. \tag{174}$$

Putting the values of A and B in (168) and making use of the inversion formula (165), the second system of stresses come out to be

$$(P_{RR})_2 = -\frac{Z_0}{2\pi^2 R(\lambda+2\mu)}\int_0^\infty\Big[A_1(\alpha)\left\{(2-2\nu+R^2\alpha^2)I_1(R\alpha)\right.$$

$$\left.-R\alpha\ I_0(R\alpha)\right\} - B_1(\alpha)\left\{I_1(R\alpha) - R\alpha\ I_0(R\alpha)\right\}\Big]\frac{\sin\alpha Z}{\Delta_1(\alpha)}\,d\alpha,$$

$$(P_{\theta\theta})_2 = -\frac{Z_0}{2\pi^2 R(\lambda+2\mu)}\int_0^\infty\Big[A_1(\alpha)\left\{(1-2\nu)R\alpha\ I_0(R\alpha)-2(1-\nu)I_1(R\alpha)\right\}$$

$$+ I_1(R\alpha)\ B_1(\alpha)\Big]\ \frac{\sin\alpha Z}{\Delta_1(\alpha)}\ d\alpha,$$

$$(P_{ZZ})_2 = \frac{Z_0}{2\pi^2(\lambda+2\mu)}\int_0^\infty\Big[A_1(\alpha)\left\{2I_0(R\alpha) + R\alpha\ I_1(R\alpha)\right\}$$

$$+ I_0(R\alpha)\ B_1(\alpha)\Big]\ \frac{\alpha\sin\alpha Z}{\Delta_1(\alpha)}\ d\alpha,$$

$$(P_{RZ})_2 = - \frac{Z_0}{2\pi^2(\lambda+2\mu)} \int_0^\infty \left[R\ \alpha\ I_0(R\alpha)A_1(\alpha) + I_1(R\alpha)B_1(\alpha)\right] \frac{\alpha\cos\alpha Z}{\Delta_1(\alpha)} d\alpha,$$

$$(P_{R\theta})_2 = 0 \quad \text{and} \quad (P_{\theta Z})_2 = 0\ , \tag{175}$$

where

$$A_1(\alpha) = -\ \{(\lambda+2\mu) + a^2\alpha^2(\lambda+\mu)\}\ \alpha^2 K_1(a\alpha)I_1(a\alpha)$$
$$- a^2\alpha^4(\lambda+\mu)\ K_0(a\alpha)\ I_0(a\alpha) + (\lambda+2\mu)\alpha^2, \tag{176}$$

$$B_1(\alpha) = -(\lambda+2\mu)\{2(1-\nu) + a^2\alpha^2\}\ \alpha^2 K_1(a\alpha)I_1(a\alpha)$$
$$-(\lambda+2\mu)a^2\alpha^4\ I_0(a\alpha)\ K_0(a\alpha) + \alpha^2\{(\lambda+2\mu)+a^2\alpha^2(\lambda+\mu)\}, \tag{177}$$

$$\Delta_1(\alpha) = \{2(1-\nu)+a^2\alpha^2\}\ \alpha^2\{I_1(a\alpha)\}^2 - a^2\alpha^4\{I_0(a\alpha)\}^2. \tag{178}$$

It may be noted that the stresses in the cylinder are

$$P_{RR} = (P_{RR})_1 + (P_{RR})_2\ ,$$
$$P_{\theta\theta} = (P_{\theta\theta})_1 + (P_{\theta\theta})_2\ ,$$
$$P_{ZZ} = (P_{ZZ})_1 + (P_{ZZ})_2\ , \tag{179}$$
$$P_{RZ} = (P_{RZ})_1 + (P_{RZ})_2\ ,$$
$$P_{R\theta} = 0 \quad \text{and } P_{\theta Z} = 0\ .$$

On putting $R = a$ in (179), P_{RR} and P_{RZ} come out to be zero, which should be the case. Also as the radius of the cylinder tends to infinity i.e. when $a \to \infty$, each of the stresses $(P_{RR})_2$, $(P_{\theta\theta})_2$, $(P_{ZZ})_2$, $(P_{RZ})_2$ is zero and the results of axial force in an infinite medium are obtained.

It will now be shown that the integrands in (175), regarded as functions of α are free from singularities. The singularities of the integrands may arise from the zeros of $\Delta_1(\alpha)$ and the poles of the numerators. It may be shown that the zeros of $\Delta_1(\alpha)$ do not exist for the range of α from 0 to ∞. The argument is as follows :

$$\Delta_1(\alpha) = 2(1-\nu)\,\alpha^2 \,\{I_1(a\alpha)\}^2 + a^2\alpha^4\left[\{I_1(a\alpha)\}^2 - \{I_o(a\alpha)\}^2\right].$$

Using the relation

$$I_m(\alpha)\, I_n(\alpha) = \left(\frac{\alpha}{2}\right)^{m+n} \sum_{\ell=0}^{\infty} \frac{\lfloor m+n+2\ell}{\lfloor \ell \,\lfloor m+\ell\, \lfloor n+\ell\, \lfloor m+n+\ell} \left(\frac{\alpha}{2}\right)^{2\ell}, \quad (180)$$

it may be seen that

$$\{I_1(a\alpha)\}^2 - \{I_o(a\alpha)\}^2 = -\sum_{\ell=0}^{\infty} \frac{\lfloor 2\ell}{(\lfloor \ell)^3 \lfloor \ell+1} \left(\frac{a\alpha}{2}\right)^{2\ell}$$

Finally, the expression for $\Delta_1(\alpha)$ may be written as

$$\Delta_1(\alpha) = - a^2\alpha^4 \sum_{\ell=0}^{\infty} \left[1 - \frac{(1+2\ell)(1-\nu)}{(1+\ell)(2+\ell)}\right] \frac{\lfloor 2\ell}{(\lfloor \ell)^3 \lfloor \ell+1} \left(\frac{a\alpha}{2}\right)^{2\ell} .$$

(181)

The expression in the square brackets of (181) is positive for all integral values of ℓ so long as $0 < \nu < \frac{1}{2}$. Therefore there does not exist any real zero of the denominator except at $\alpha = 0$. However, it may be noted that in the limit as $\alpha \to 0$; the integrands in (175) either tend to zero or to some finite limit but do not become infinite.

The integrands in (175) contain the functions $K_0(a\alpha)$ and $K_1(a\alpha)$ which are singular at $\alpha = 0$. But these are multiplied by such functions that when $\alpha \to 0$ and the limit of the product is taken, it tends to zero or some finite limit.

Making use of the following asymptotic expansions ((23))

$$K_n(\alpha) = \sqrt{\frac{\pi}{2\alpha}}\, e^{-\alpha} \left\{ 1 + \frac{(4n^2-1^2)}{\lfloor 1 \, (8\alpha)} + \frac{(4n^2-1^2)(4n^2-3^2)}{\lfloor 2 \, (8\alpha)^2} + \dots \infty \right\}$$

$$\alpha \to \infty, \qquad (182)$$

and

$$I_n(\alpha) = \frac{1}{\sqrt{2\pi\alpha}} e^{\alpha} \left\{ 1 - \frac{4n^2-1^2}{\lfloor 1 \, (8\alpha)} + \frac{(4n^2-1^2)(4n^2-3^2)}{\lfloor 2 \, (8\alpha)^2} \cdots \infty \right\},$$

$$\alpha \to \infty , \qquad (183)$$

the order of convergence of each of the integrands in (175), may be given as

$$O\left(\alpha^{5/2} \; e^{(R/a - 2)\alpha} \right) \text{ as } \alpha \to \infty . \qquad (184)$$

The integrals in (175) have been evaluated numerically for some cases. The method of the numerical solution of the integrals given in (175) may be devided into two parts. The first one is that of calculating the modified Bessel functions $I_o(\alpha)$, $I_1(\alpha)$, $K_o(\alpha)$ and $K_1(\alpha)$. For calculating these functions, the series expansions given below have been employed ((23)).

$$I_n(\alpha) = \sum_{r=o}^{\infty} \frac{1}{\Gamma(r+1)\;\Gamma(n+r+1)} \left(\frac{\alpha}{2} \right)^{n+2r}, \qquad (185)$$

$$K_n(\alpha) = (-1)^{n+1} I_n(\alpha) \log\left(\frac{\alpha}{2} \right) + \frac{1}{2} \sum_{m=o}^{n-1} (-1)^m \left(\frac{\alpha}{2} \right)^{2m-n} \frac{\lfloor n-m-1}{\lfloor m}$$
$$+ \frac{1}{2}(-1)^n \sum_{m=o}^{\infty} \frac{\left(\frac{\alpha}{2} \right)^{n+2m}}{\lfloor m \; \lfloor n+m} \left[\psi(n+m+1) + \psi(m+1) \right] . \qquad (186)$$

For large values of the argument, asymptotic expansions for $K_0(\alpha)$ and $K_1(\alpha)$ given in (183) and (184) have been used. The values of $I_0(\alpha)$, $I_1(\alpha)$, $K_0(\alpha)$ and $K_1(\alpha)$ calculated for different values of the argument were found to be the same as given in ((24)).

The second part is the evaluation of the infinite integrals. The presence of oscillatory functions sin αZ and cos αZ , suggests the use of L.N.G. Filon's method . However as stated in chapter VIII, if the interval length is small, one can employ even Simpson's rule .

The upper limit of the infinite integral may be taken as some finite value of α which is to be chosen suitably keeping in mind the result given in (184). The infinite integrals were truncated at $\alpha = 20$ and $\alpha = 25$ and the difference in the values of the integrals for these two truncation limits is insignificant. The interval length was taken to be 0.1 . The ratio z/a takes values 0.0, 1.0, 2.0, 3.0, 4.0 and 5.0 . The value of Poisson ratio is taken as 0.25 .

It may be noted that the stresses in the cylinder are the sum of two systems of stresses ; the first and the second stress systems as given in (179). The first system of stresses are not complicated and can be

calculated with a fair degree of accuracy. The second system of stresses involve infinite integrals. Filon's method and Simpson's rule both were employed for this finite interval and the results obtained by both of these methods agree closely.

It is interesting to note the behaviour of P_{RR}, P_{RZ}, $P_{\theta\theta}$ and P_{ZZ} as $Z \to \infty$. It may be seen that P_{RR}, P_{RZ} and $P_{\theta\theta}$ vanish as $Z \to \infty$. But P_{ZZ} attains a constant value as $Z \to \infty$. The integral for P_{ZZ} has the form

$$\int_0^\infty f(\alpha) \frac{\sin \alpha Z}{\alpha} d\alpha ,$$

which has the value $\frac{\pi}{2} f(0)$ as $Z \to \infty$. The analytical value of P_{ZZ} is $-Z_0 / 2\pi a^2$ as $Z \to \infty$. The numerical value of P_{ZZ} is $-0.49989 \times Z_0/\pi a^2$ which is attained approximately for all $Z/a > 4$.

The graphs of stresses P_{RR}, P_{RZ}, $P_{\theta\theta}$ and P_{ZZ} in the cylinder have been drawn for all the cases mentioned above and are given in figures 2, 4, 6 and 8 in the Appendix following this chapter. A comparison of these stresses has been done with the stresses due to a point force along the Z axis at the origin in the infinite medium in figures 3, 5, 7 and 9.

APPENDIX TO CHAPTER IX

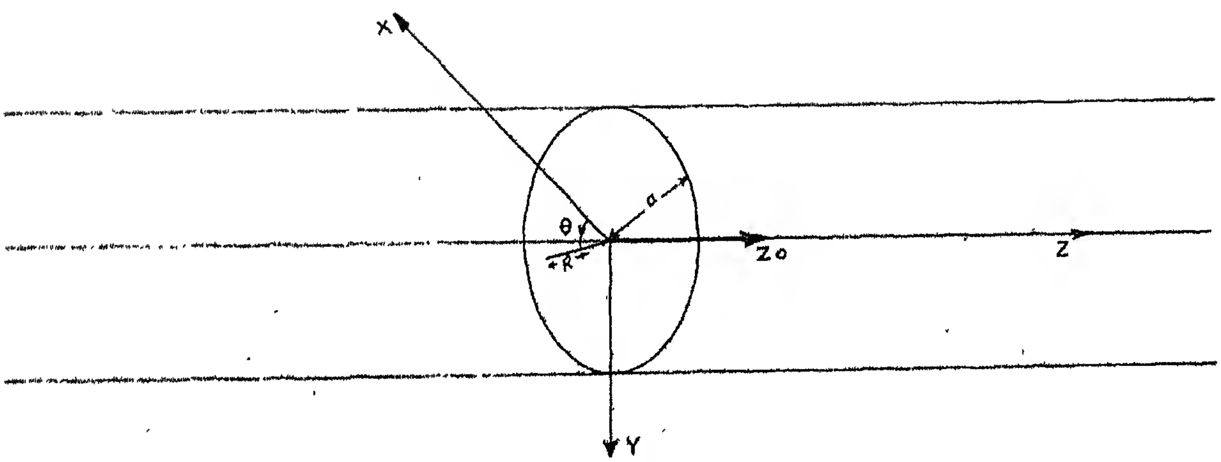


FIG. I. THE POINT FORCE Zo AND THE COORDINATE SYSTEM (R, θ, Z).

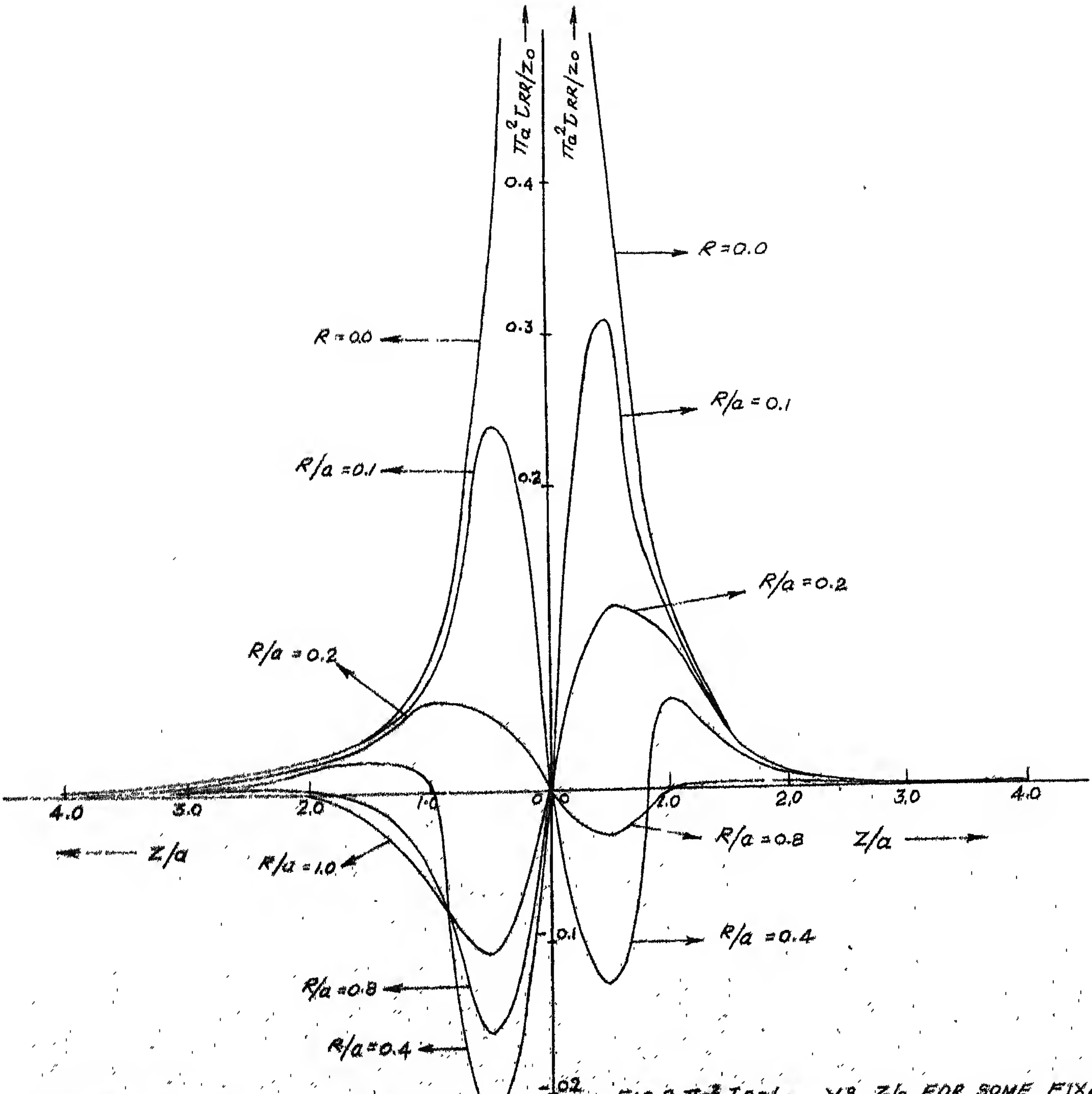


FIG. 3. $\pi a^2 \tau_{RR}/Z_0$ VS Z/a FOR SOME FIXED R/a's FOR A POINT FORCE IN Z-DIRECTION IN AN INFINITE MEDIUM. $\sigma = .25$

FIG. 2. $\pi a^2 \tau_{RR}/Z_0$ VS Z/a FOR SOME FIXED R/a's FOR AN AXIAL POINT FORCE IN A CYLINDER OF RADIUS a. $\sigma = .25$

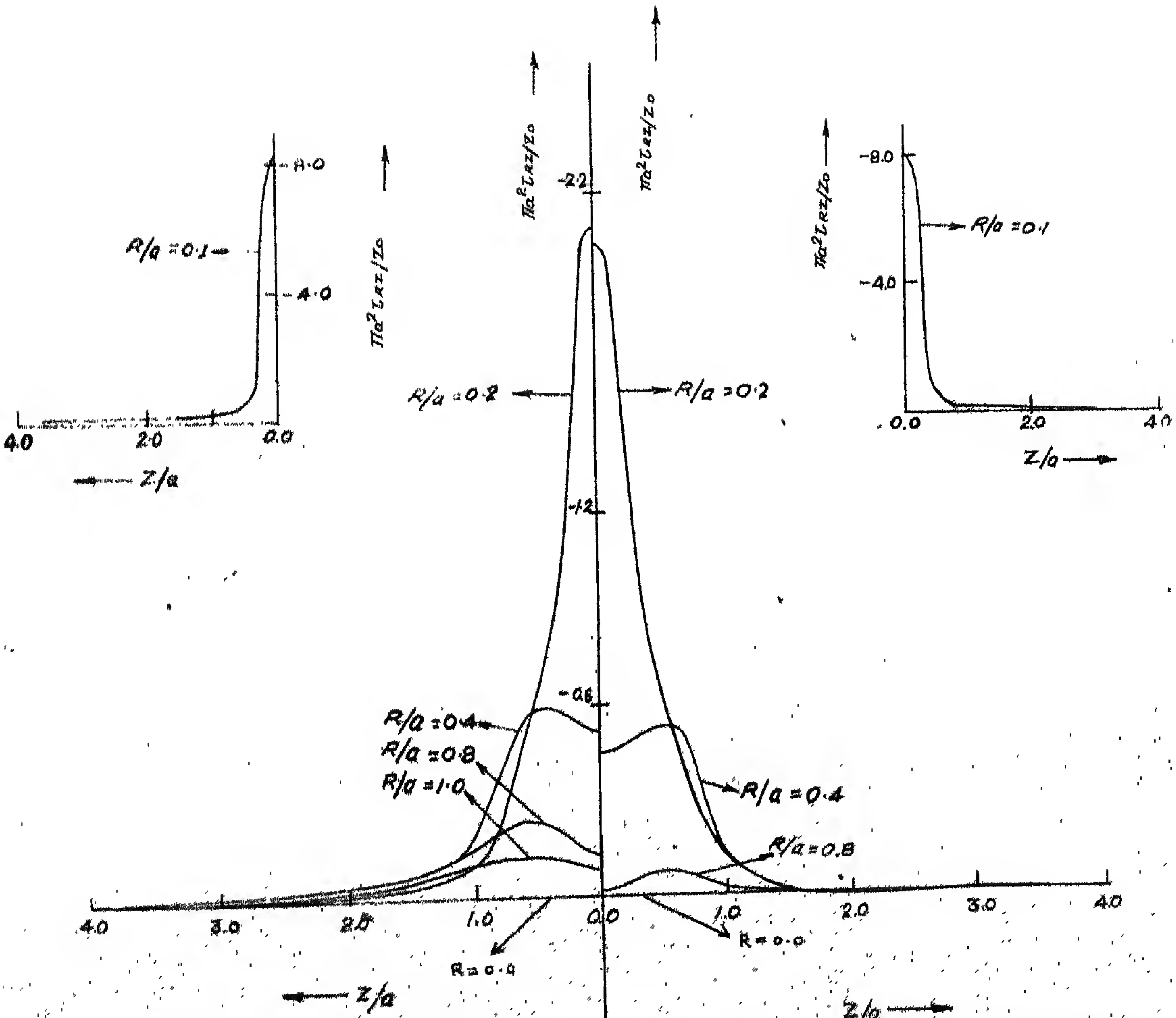


FIG. 5. $\pi a^2 \tau_{RZ}/Z_0$ VS Z/a FOR SOME FIXED R/a'S FOR A POINT FORCE IN Z-DIRECTION IN AN INFINITE MEDIUM. $\sigma = \cdot 25$

FIG. 4. $\pi a^2 \tau_{RZ}/Z_0$ VS Z/a FOR SOME FIXED R/a'S FOR AN AXIAL POINT FORCE IN A CYLINDER OF RADIUS a. $\sigma = \cdot 25$

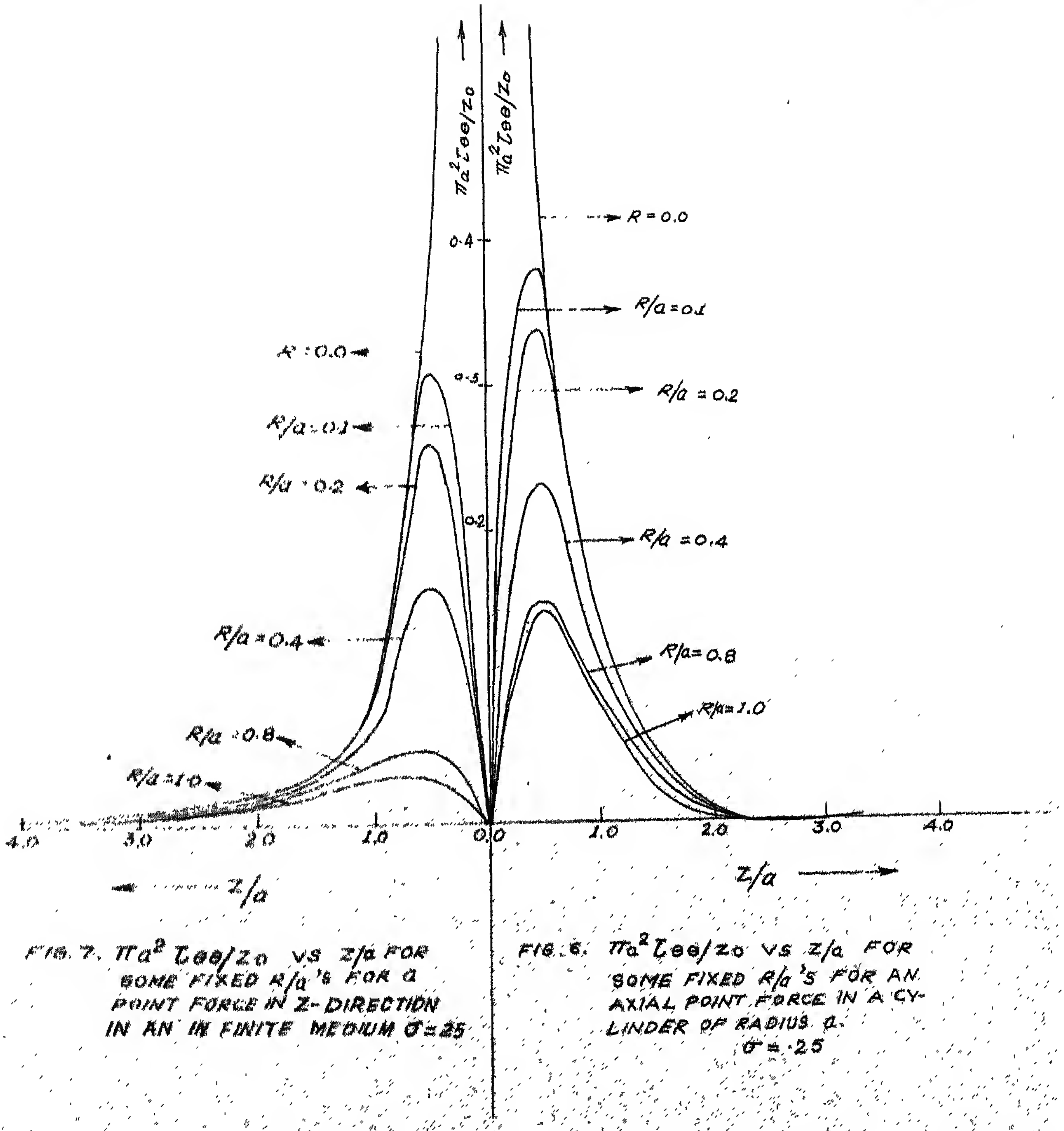


FIG. 7. $\pi a^2 \tau_{\theta\theta}/Z_o$ VS Z/a FOR SOME FIXED R/a'S FOR A POINT FORCE IN Z-DIRECTION IN AN INFINITE MEDIUM $\sigma = .25$

FIG. 6. $\pi a^2 \tau_{\theta\theta}/Z_o$ VS Z/a FOR SOME FIXED R/a'S FOR AN AXIAL POINT FORCE IN A CYLINDER OF RADIUS a. $\sigma = .25$

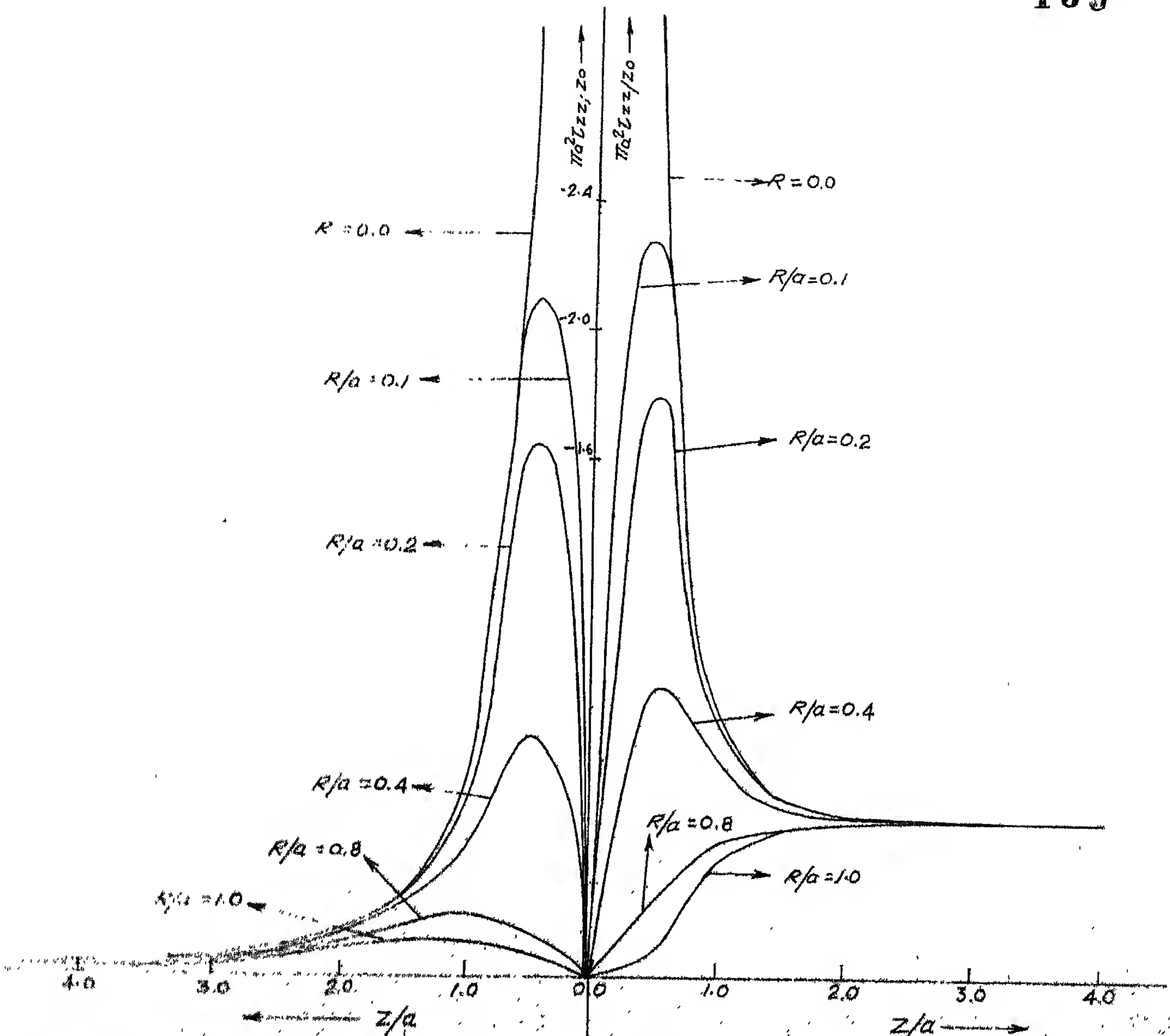


FIG. 9. $\pi a^2 \tau_{zz}/z_0$ VS Z/a FOR SOME FIXED R/a'S FOR A POINT FORCE IN Z-DIRECTION IN AN INFINITE MEDIUM $\sigma = .25$

FIG. 8. $\pi a^2 \tau_{zz}/z_0$ VS Z/a FOR SOME FIXED R/a'S FOR AN AXIAL POINT FORCE IN A CYLINDER OF RADIUS a. $\sigma = .25$

CHAPTER X

POINT FORCE IN AN INFINITE ELASTIC WEDGE

In this chapter the following plane strain problem is considered.

Suppose there is an infinite elastic solid, which is bounded by the two planes $\theta = \pm\theta_0$ $(R, \theta, Z$ being the cylindrical polar coordinates). Thus the edge of the wedge is along Z axis. In Cartesian coordinates, therefore, the x axis is the axis of symmetry and the y axis is perpendicular to Z and x axes. Note that z axis in Cartesian coordinates is coincident with Z axis in cylindrical polar coordinates. The system is supposed to be in a state of plain strain. Hence we might consider only a plane cross-section perpendicular to the Z axis. We shall henceforth work in (x,y) plane only and consider the stresses P_{xx}, P_{xy}

and P_{yy} or their equivalent in polar coordinates.

It may be remarked at this stage, that the problems worked in the previous chapters were axi-symmetric. But in the present problem the point force is not necessarily acting along the axis of symmetry. Let a point force (X_o, Y_o) act in the wedge at a point whose Cartesian coordinates are (h,k) (Figure 2 p.139). The solution to this problem has been achieved by the superposition of two stress systems. The first system of stresses is that due to a point force in an infinite two-dimensional elastic plane. This we have named as the first stress system. On the faces of the wedge $\theta = \pm \theta_o$, normal and shearing stresses may be calculated from the first stress system.

The second system of stresses are obtained by solving the following problem : consider a wedge whose faces are bounded by the lines $\theta = \pm \theta_o$ (note that we are working a two-dimensional problem). Apply on the faces of the wedge $P_{\theta\theta}$ and $P_{\theta r}$ which are equal in magnitude but opposite in sign to those obtained from first stress system at $\theta = \pm \theta_o$. Stresses would develop in the wedge. These stresses are the second system of stresses. The superposition of these two stress systems would give the stresses due to a point force inside the

wedge, with zero tractions on the boundary.

The first system of stresses i.e. when the point force (X_o, Y_o) acts at a point (h,k) are given by (114). These stresses in polar coordinates (r,θ) $(x = r\cos\theta,\ y = r\sin\theta)$ may be easily derived from (114) by transforming the stress system in polar coordinates. They are

$$(P_{rr})_1 = -\frac{X_o}{4\pi(1-\nu)}\left[(1-2\nu)\frac{\{r\cos\theta - h\cos2\theta - k\sin2\theta\}}{R_1^2}\right.$$

$$\left.+\frac{2(r\cos\theta-h)\{r-h\cos\theta - k\sin\theta\}^2}{R_1^4}\right]$$

$$-\frac{Y_o}{4\pi(1-\nu)}\left[(1-2\nu)\frac{\{r\sin\theta - h\sin2\theta + k\cos2\theta\}}{R_1^2}\right.$$

$$\left.+\frac{2(r\sin\theta-k)\{r-h\cos\theta - k\sin\theta\}^2}{R_1^4}\right],$$

$$(P_{\theta\theta})_1 = -\frac{X_o}{4\pi(1-\nu)}\left[\frac{(1-2\nu)\{-r\cos\theta + h\cos2\theta + k\sin2\theta\}}{R_1^2}\right.$$

$$\left.+\frac{2(r\cos\theta-h)(k\cos\theta - h\sin\theta)^2}{R_1^4}\right].$$

$$-\frac{Y_o}{4\pi(1-\nu)}\left[(1-2\nu)\frac{\{-r\sin\theta + h\sin2\theta - k\cos2\theta\}}{R_1^2}\right.$$

$$+ \frac{2(r\sin\theta - k)(k\cos\theta - h\sin\theta)^2}{R_1^4}\Bigg],$$

$$(P_{r\theta})_1 = -\frac{X_o}{4\pi(1-\nu)}\Bigg[\frac{(1-2\nu)\{-r\sin\theta + h\sin 2\theta - k\cos 2\theta\}}{R_1^2}$$

$$+ \frac{2r^2\cos\theta(h\sin\theta - k\cos\theta)}{R_1^4}$$

$$+ \frac{2r\{2hk\cos^3\theta + \sin\theta(k^2\cos^2\theta - h^2 - h^2\cos^2\theta)\}}{R_1^4}$$

$$- \frac{2h\{hk\cos 2\theta + \sin\theta\cos\theta(k^2-h^2)\}}{R_1^4}\Bigg]$$

$$- \frac{Y_o}{4\pi(1-\nu)}\Bigg[\frac{(1-2\nu)\{r\cos\theta - h\cos 2\theta - k\sin 2\theta\}}{R_1^2}$$

$$+ \frac{2r^2\sin\theta(h\sin\theta - k\cos\theta)}{R_1^4}$$

$$+ \frac{2r\{\cos\theta(k^2+k^2\sin^2\theta - h^2\sin^2\theta) - 2hk\sin^3\theta\}}{R_1^4}$$

$$- \frac{2k\{hk\cos 2\theta + \sin\theta\cos\theta(k^2-h^2)\}}{R_1^4}\Bigg], \qquad (187)$$

where $R_1^2 = r^2 + h^2 + k^2 - 2r(h \cos\theta + k \sin\theta)$.

The subscript 1 refers to the first system of stresses. From the above expressions of stresses given in (187) , $(P_{\theta\theta})_1$ and $(P_{r\theta})_1$ at $\theta = \pm\,\theta_0$ may be easily calculated. Now apply on the faces of the wedge normal and shearing stresses $P_{\theta\theta}$ and $P_{r\theta}$, equal in magnitude and opposite in sign to those obtained from (187) by putting $\theta = \pm\,\theta_0$.

Thus, we apply $(P_{\theta\theta})_2$ and $(P_{r\theta})_2$ on the faces of the wedge such that

$$\left\{(P_{\theta\theta})_2\right\}_{\theta = \pm\theta_0} = -\left\{(P_{\theta\theta})_1\right\}_{\theta = \pm\theta_0} \tag{188}$$

and $$\left\{(P_{r\theta})_2\right\}_{\theta = \pm\theta_0} = -\left\{(P_{r\theta})_1\right\}_{\theta = \pm\theta_0} . \tag{189}$$

The solution of this auxiliary problem will now be described.

The equations of equilibrium in polar coordinates in the absence of body forces and their solution in terms of a function $\phi(r,\theta)$ which satisfies the biharmonic equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right) = 0 \tag{190}$$

is given in details in ((17)). It is sufficient to mention here that the stresses may be expressed directly in terms of the function ϕ such that

$$P_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} ,$$

$$P_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} ,$$

$$P_{r\theta} = - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta} \right) . \tag{191}$$

Now if $\bar{\phi}$ is defined as

$$\bar{\phi} = \int_0^\infty r^{s-3}\,\phi\, dr , \tag{192}$$

then it may be seen that the solution of (190) is

$$\bar{\phi} = A \sin(s-2)\theta + B \cos(s-2)\theta + C \sin s\theta + D \cos s\,\theta . \tag{193}$$

This solution in (193) may also be obtained from the solution given in ((17)) by putting s-2 for s (s has been replaced by s-2 for convenience).

Multiplying each of the equations in (191) by r^{s-1} and integrating with respect to r from 0 to ∞ and using (193), we get

$$\overline{(P_{rr})_2} = - A(s-2)(s-1)\sin(s-2)\theta - B(s-2)(s-1)\cos(s-2)\theta$$
$$- C(s+2)(s-1) \sin s\theta - D(s+2)(s-1) \cos s\theta ,$$

$$\overline{(P_{\theta\theta})_2} = (s-2)(s-1)\{ A \sin(s-2)\theta + B \cos(s-2)\theta + C \sin s\theta$$
$$+ D \cos s\theta\} ,$$

$$\overline{(P_{r\theta})_2} = (s-2)(s-1)\{ A \cos(s-2)\theta - B \sin(s-2)\theta\}$$
$$+ s (s-1) \{C \cos s\theta - D \sin s\theta\} , \quad (194)$$

where bar denotes the Mellin transform of a function as defined in (126) and the subscript two has been used for the stresses in (194) as this will be the second stress system.

Multiplying the boundary conditions in (188) and (189) by r^{s-1} and integrating with respect to r from 0 to ∞, we get

$$\int_0^\infty r^{s-1}\{(P_{\theta\theta})_2\}_{\theta= \pm \theta_o} dr = - \int_0^\infty r^{s-1} \{(P_{\theta\theta})_1\}_{\theta= \pm \theta_o} dr$$

$$\text{or} \quad \overline{\{(P_{\theta\theta})_2\}}_{\theta = \pm \theta_o} = - \overline{\{(P_{\theta\theta})_1\}}_{\theta = \pm \theta_o} \quad (195)$$

and

$$\int_0^\infty r^{s-1}\{(P_{r\theta})_2\}_{\theta=\pm\theta_o} dr = - \int_0^\infty r^{s-1}\{(P_{r\theta})_1\}_{\theta=\pm\theta_o} dr$$

Or
$$\overline{\{(P_{r\theta})_2\}}_{\theta=\pm\theta_o} = - \overline{\{(P_{r\theta})_1\}}_{\theta=\pm\theta_o} \tag{196}$$

The infinite integrals occurring on the right hand sides of (195) and (196) may be evaluated using the result

$$\int_0^\infty \frac{r^{s-1}\, dr}{(r^2+2ar\cos\theta + a^2)^2} = - \pi a^{s-2} \operatorname{cosec}\theta \operatorname{cosec}(\pi s) \cdot \sin(s-1)\theta \tag{197}$$

$$0 < Re(s) < 2, \quad a > 0, \quad -\pi < \theta < \pi,$$

given in ((18)), after suitable modifications.

Let the right hand sides of (195) and (196) be denoted by $f_{22}(s,\theta_o)$ and $f_{11}(s,\theta_o)$ respectively, so that

$$\overline{\{(P_{\theta\theta})_2\}}_{\theta=\theta_o} = f_{22}(s,\theta_o),$$

$$\overline{\{(P_{\theta\theta})_2\}}_{\theta=-\theta_o} = f_{22}(s,-\theta_o),$$

$$\overline{\{(P_{r\theta})_2\}}_{\theta=\theta_o} = f_{11}(s,\theta_o),$$

$$\left\{(P_{r\theta})_2\right\}_{\theta = -\theta_0} = f_{11}(s, -\theta_0) \ . \tag{198}$$

The expressions of $f_{11}(s, \theta_0)$, $f_{11}(s,-\theta_0)$, $f_{22}(s,\theta_0)$ and $f_{22}(s, -\theta_0)$ in real and imaginary parts are given in the Appendix II following this chapter .

It may be seen that all the integrals on the right hand sides in (195) and (196) are valid for $0 < \mathrm{Re}(s) < 1$. With the help of (194) and (198) , we can determine the four constants A, B, C and D. They are

$$A = \frac{\left\{f_{11}(s,\theta_0)+ f_{11}(s,-\theta_0)\right\}(s-2)\sin s\theta_0 - \left\{f_{22}(s,\theta_0) - f_{22}(s,-\theta_0)\right\} s \cos s\theta_0}{2(s-2)(s-1)\left\{(s-1) \sin 2\theta_0 - \sin 2(s-1)\theta_0\right\}} , \tag{199}$$

$$B = \frac{\left\{f_{11}(s,-\theta_0)- f_{11}(s,\theta_0)\right\}(s-2) \cos s\theta - \left\{f_{22}(s,\theta_0) + f_{22}(s,-\theta_0)\right\} s \sin s\theta_0}{2(s-2)(s-1)\left\{(1-s)\sin 2\theta_0 - \sin 2(s-1)\theta_0\right\}} , \tag{200}$$

$$C = -\frac{\left\{f_{11}(s,\theta_0)+ f_{11}(s,-\theta_0)\right\}\sin(s-2)\theta_0 - \left\{f_{22}(s,\theta_0) - f_{22}(s,-\theta_0)\right\}\cos(s-2)\theta_0}{2(s-1)\left\{(s-1)\sin 2\theta_0 - \sin 2(s-1)\theta_0\right\}} , \tag{201}$$

$$D = -\frac{\{f_{11}(s,-\theta_o) - f_{11}(s,\theta_o)\}\cos(s-2)\theta_o - \{f_{22}(s,\theta_o) + f_{22}(s,-\theta_o)\}\sin(s-2)\theta_o}{2(s-1)\{(1-s)\sin 2\theta_o - \sin 2(s-1)\theta_o\}} . \quad (202)$$

Substituting these values of the constants A, B, C and D in (194) and inverting (194) with the help of the inversion formula given in (127), we get

$$(P_{rr})_2 = -\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Big[\frac{C_1}{\Delta_1(s,\theta_o)}\{(s-2)\sin s\theta_o \sin(s-2)\theta$$

$$-(s+2)\sin s\theta \sin(s-2)\theta_o\} - \frac{C_2}{\Delta_1(s,\theta_o)}\{s\cos s\theta_o$$

$$\cdot\sin(s-2)\theta - (s+2)\sin s\theta\cos(s-2)\theta_o\} + \frac{C_1'}{\Delta_2(s,\theta_o)}$$

$$\cdot\{(s-2)\cos s\theta_o\cos(s-2)\theta - (s+2)\cos s\theta\cos(s-2)\theta_o\}$$

$$-\frac{C_2'}{\Delta_2(s,\theta_o)}\{s\sin s\theta_o\cos(s-2)\theta - (s+2)\cos s\theta$$

$$\sin(s-2)\theta_o\}\Big]\,ds ,$$

$$(P_{\theta\theta})_2 = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{C_1}{\Delta_1(s,\theta_o)}(s-2)\{\sin s\theta_o \sin(s-2)\theta -$$

$$-\sin(s-2)\theta_o\sin s\theta\} -$$

$$- \frac{C_2}{\Delta_1(s,\theta_0)} \{ s \cos s\theta_0 \sin(s-2)\theta - (s-2)\cos(s-2)\theta_0 \sin s\theta \}$$

$$+ \frac{C_1'}{\Delta_2(s,\theta_0)} (s-2)\{\cos s\theta_0 \cos(s-2)\theta - \cos s\theta \cos(s-2)\theta_0\}$$

$$- \frac{C_2'}{\Delta_2(s,\theta_0)} \{ s \sin s\theta_0 \cos(s-2)\theta - (s-2)\cos s\theta \sin(s-2)\theta_0 \} \Bigg] ds,$$

(203)

$$(P_{r\theta})_2 = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \Bigg[\frac{C_1}{\Delta_1(s,\theta_0)} \{(s-2)\sin s\theta_0 \cos(s-2)\theta$$

$$- s \cos s\theta \sin(s-2)\theta_0 \} - \frac{C_2}{\Delta_1(s,\theta_0)} s\{\cos s\theta_0 \cos(s-2)\theta$$

$$- \cos s\theta \cos(s-2)\theta_0 \} + \frac{C_1'}{\Delta_2(s,\theta_0)} \{-(s-2)\cos s\theta_0 \sin(s-2)\theta$$

$$+ s \sin s\theta \cos(s-2)\theta_0 \} + \frac{C_2'}{\Delta_2(s,\theta_0)} s\{\sin s\theta_0 \sin(s-2)\theta$$

$$- \sin s\theta \sin(s-2)\theta_0 \} \Bigg] ds ,$$

where

$$\Delta_1(s,\theta_0) = 2\{s \sin 2\theta_0 - 2 \sin s\theta_0 \cos(s-2)\theta_0\}, \qquad (204)$$

$$\Delta_2(s,\theta_0) = 2\left\{- s \sin 2\theta_0 - 2 \cos s\theta_0 \sin(s-2)\theta_0\right\}, \tag{205}$$

$$\begin{aligned} C_1 &= f_{11}(s,\theta_0) + f_{11}(s,-\theta_0) \quad , \\ C_1' &= f_{11}(s,-\theta_0) - f_{11}(s,\theta_0) \quad , \\ C_2 &= f_{22}(s,\theta_0) - f_{22}(s,-\theta_0) \quad , \\ C_2' &= f_{22}(s,\theta_0) + f_{22}(s,-\theta_0) \quad . \end{aligned} \tag{206}$$

The integrals in (203) are not affected by the particular choice of γ, as long as $\mathrm{Re}(s)$ $(= \gamma)$ remains within the limits $0 < \mathrm{Re}(s) < 1$. The choice of $\gamma = 1/2$, which in turn implies

$$s = \frac{1}{2} + i\alpha \ , \quad -\infty < \alpha < \infty \quad , \tag{207}$$

is suggested by the fact that the functions occurring in the integrands of (203) are simplified to a great extent.

In order to establish the validity of the choice of γ , we have yet to show that the integrands in (203), regarded as functions of s, are free from singularities, if (207) holds. The singularities of the integrands may arise from the zeros of $\sin s\pi$ which occur in the denominators of $f_{11}(s,\theta_0)$, $f_{11}(s,-\theta_0)$, $f_{22}(s,\theta_0)$ and $f_{22}(s,-\theta_0)$ and the zeros of $\Delta_1(s,\theta_0)$ and $\Delta_2(s,\theta_0)$. It

may be noted that sin $s\pi$ does not vanish for the values of s given in (207). $\Delta_1(s,\theta_0)$ and $\Delta_2(s,\theta_0)$ may be written as

$$\Delta_1(s,\theta_0) = -\sin 2\theta_0 + 2\sin\theta_0 \cosh 2\alpha\theta_0 - 2i(\cos\theta_0 \sinh 2\alpha\theta_0 - \alpha \sin 2\theta_0) ,$$

$$\Delta_2(s,\theta_0) = -(\sin 2\theta_0 + \sin\theta_0 \cosh 2\alpha\theta_0) + 2i(\alpha \sin 2\theta_0 + \cos\theta_0 \cdot \sinh 2\alpha\,\theta_0) .$$

It may be seen from the above expressions that $\Delta_1(s,\theta_0)$ and $\Delta_2(s,\theta_0)$ are not zero for positive real values of α and $\theta_0 > 0$.

We now put $s = \frac{1}{2} + i\alpha$ in (203) and separate the integrals in real and imaginary parts. The real and imaginary parts are found to be even and odd functions of α, respectively. Hence, the contribution of imaginary part is zero and we obtain the complementary stresses as given below.

$$(P_{rr})_2 = -\frac{1}{\pi r^{1/2} r_o^{7/2}} \int_0^\infty (F_{11}\cos m_1\alpha - F_{22}\sin m_1\alpha)\,d\alpha ,$$

$$(P_{\theta\theta})_2 = \frac{1}{\pi r^{1/2} r_o^{7/2}} \int_0^\infty (F_{33}\cos m_1\alpha - F_{44}\sin m_1\alpha)\,d\alpha ,$$

$$(P_{r\theta})_2 = \frac{1}{\pi r^{1/2} r_o^{7/2}} \int_0^\infty (F_{55}\cos m_1\alpha - F_{66}\sin m_1\alpha)\,d\alpha , \tag{208}$$

where

$$m_1 = \log (r_0/r) , \quad r_0^2 = h^2 + k^2 ;$$

the quantities F_{11}, F_{22}, F_{33}, F_{44}, F_{55} and F_{66} are given in the Appendix II to this chapter.

Finally, the stresses in the wedge due to the point force are given by

$$P_{rr} = (P_{rr})_1 + (P_{rr})_2 ,$$

$$P_{\theta\theta} = (P_{\theta\theta})_1 + (P_{\theta\theta})_2 , \tag{209}$$

$$P_{r\theta} = (P_{r\theta})_1 + (P_{r\theta})_2 ,$$

where $(P_{rr})_1$ etc. and $(P_{rr})_2$ etc. are given by (187) and (208) respectively.

It may be verified that on the faces of the wedge $\theta = \pm \theta_0$, the normal stress $P_{\theta\theta}$ and shearing stress $P_{r\theta}$ vanish, as they should. On putting $\theta_0 = \pi/2$ in (209) , the results due to a point force in the half plane are obtained. It may be remarked that while finding the second system of stresses in the wedge another problem of that of prescribed tractions on the faces of the wedge is also solved.

The integrals in (208) have been evaluated numerically for some cases and the numerical evaluation and discussion is given in the Appendix I to this chapter.

APPENDIX I TO CHAPTER X

The integrals in (208) are all convergent and if each of the integrands in (208) is denoted by $f(\alpha,\theta,\theta_0)$, then

$$f(\alpha,\ \theta,\ \theta_0) = 0\ \{\ \alpha\ (e^{\alpha(\theta-\theta_0-\theta_1)} + e^{\alpha(\theta-\theta_0-\theta_2)}\ )\},$$

where $\cos\theta_1 = \dfrac{h\ \cos\theta_0+k\ \sin\theta_0}{(h^2+k^2)^{1/2}}$ and $\cos\theta_2 = \dfrac{h\ \cos\theta_0-k\sin\theta_0}{(h^2+k^2)^{1/2}}$.

The integrals in (208) have been evaluated numerically for some cases. The infinite integrals may be truncated at some finite value of α, bearing in mind the order of convergence given above which depends upon α, the point of application of the force and the semiangle of the wedge. The semiangle of the wedge θ_0 has been taken as $\pi/2$ and $\pi/4$ and the point of application of the force (X_0,Y_0) is taken as (1,0). Two cases, when the point force is acting along the axis of symmetry and when it is acting perpendicular to the axis, have been considered. The upper limit of α is taken as 25. Poisson ratio is taken as 0.25. The methods employed for the numerical evaluation of the integrals are the same as described in previous chapters.

Boundary conditions were satisfied numerically also. It was felt that if the point of application of the

force is very near the vertex of the wedge, the stresses should not differ substantially from the stresses when the point force acts at the vertex of the wedge. The solution of the problem when a point force acts at the vertex of the wedge is given in Love's book ((16)). Stresses were calculated for our problem when a point force along the axis acts at a point 0.01 on the axis of the wedge of semiangle $\pi/4$. These stresses were compared with the stresses due to an axial point force acting at the vertex of the wedge of semiangle $\pi/4$ which can be calculated from the results given in ((16)). This comparison of the two systems of stresses so obtained for some values of r and θ is shown in Table 1 p. 177 . As expected, the stresses do not differ significantly and the maximum difference is 0.103%. This gives a further check on the calculations. Numerical values of stresses P_{rr}, $P_{\theta\theta}$ and $P_{r\theta}$ when a point force acts at (1,0) perpendicular to the x axis in a half plane are given in Table 2 , p. 178 ; r varies from 0.5 to 7.5 with an increment of 1.0 and the values given to θ are θ_0 and 0 .

Variation of stresses P_{rr}, $P_{\theta\theta}$ and $P_{r\theta}$ is shown in the form of graphs from 1 through 6 which are given in this Appendix from p. 179 to 184 . The semiangle of the wedge is $\pi/4$ and the point of application of the force is

TABLE 1

POINT FORCE = $(X_o, 0)$.

POINT OF APPLICATION OF THE FORCE = (0.01, 0).

$\theta_o = \pi/4$; $\nu = 0.25$.

$P_{\theta\theta}$ and $P_{r\theta}$ are zero each.

θ	r	P_{rr}/X_o In present problem.	P_{rr}/X_o In solution given in Love.
0	0.1	-7.7875	-7.7797
	0.2	-3.8893	-3.8898
	0.3	-2.5930	-2.5932
	0.4	-1.9448	-1.9449
$\pi/8$	0.1	-7.1868	-7.1875
	0.2	-3.5936	-3.5937
	0.3	-2.3958	-2.3958
	0.4	-1.7969	-1.7969
$\pi/4$	0.1	-5.5040	-5.5011
	0.2	-2.7517	-2.7505
	0.3	-1.8340	-1.8337
	0.4	-1.3754	-1.3753

TABLE 2

POINT FORCE = $(0, Y_o * 4\pi(1-\nu))$

POINT OF APPLICATION OF THE FORCE = (1,0)

$\theta_o = \pi/2$, $\nu = 0.25$

$P_{\theta\theta}$ is zero identically.

θ	r	P_{rr}/Y_o	$P_{r\theta}/Y_o$
	0.5	0.0000	0.2963
	1.5	0.0000	-1.3440
	2.5	0.0000	-0.5053
$\theta = 0$	3.5	0.0000	-0.2919
	4.5	0.0000	-0.1932
	5.5	0.0000	-0.1379
	6.5	0.0000	-0.1036
	7.5	0.0000	-0.0807
	0.5	0.1600	0.0000
	1.5	-1.6331	0.0000
	2.5	-1.6885	0.0000
	3.5	-1.4254	0.0000
$\theta = \pi/2$	4.5	-1.1909	0.0000
	5.5	-1.0109	0.0000
	6.5	-0.8739	0.0000
	7.5	-0.7677	0.0000

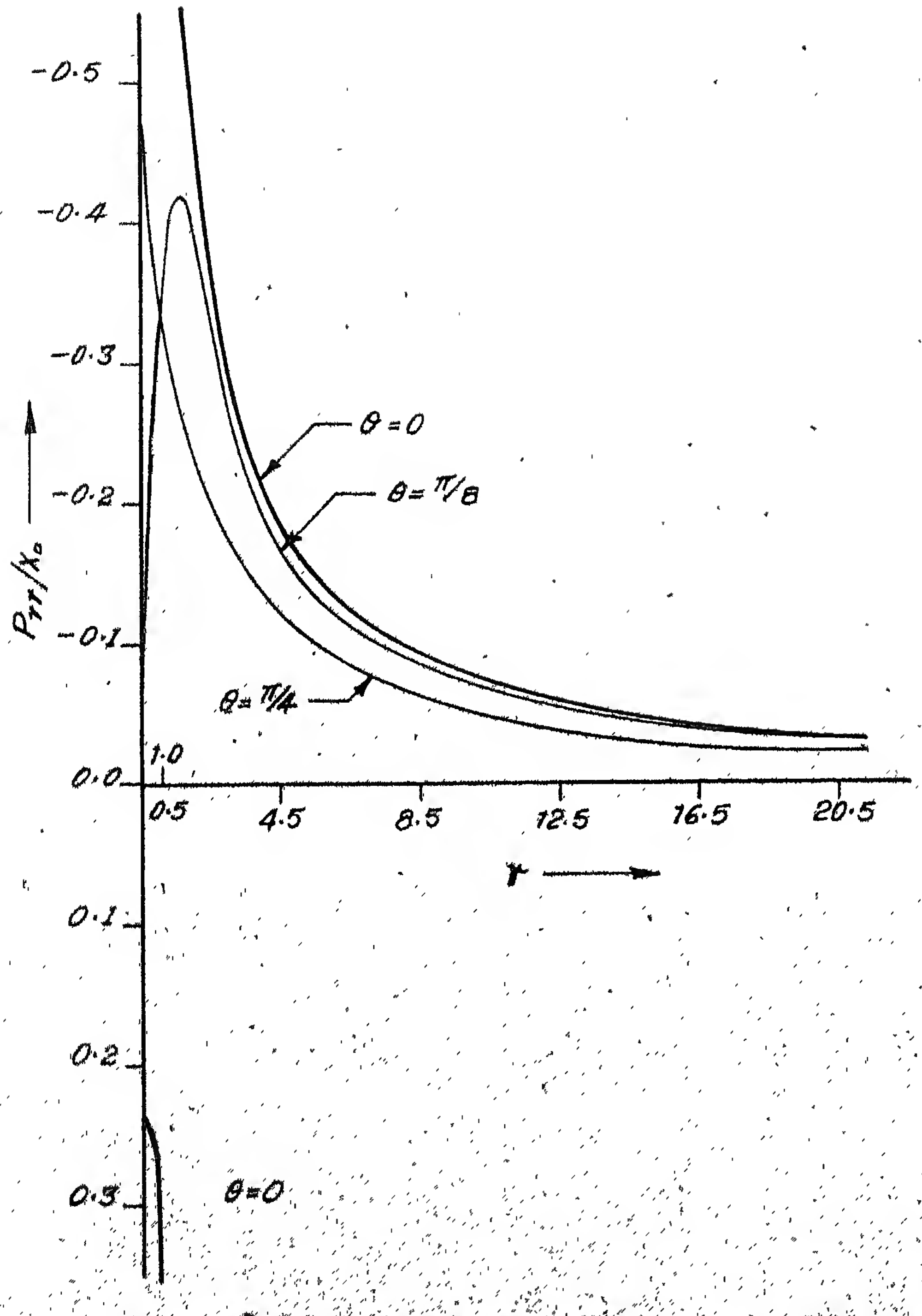


FIG.-1- P_{rr}/X_0 VS r FOR SOME FIXED θ'S FOR AN AXIAL FORCE X_0 IN AN INFINITE WEDGE OF SEMIANGLE $\pi/4$. POINT OF APPLICATION OF FORCE IS (1,0).
POISSON RATIO = 0.25

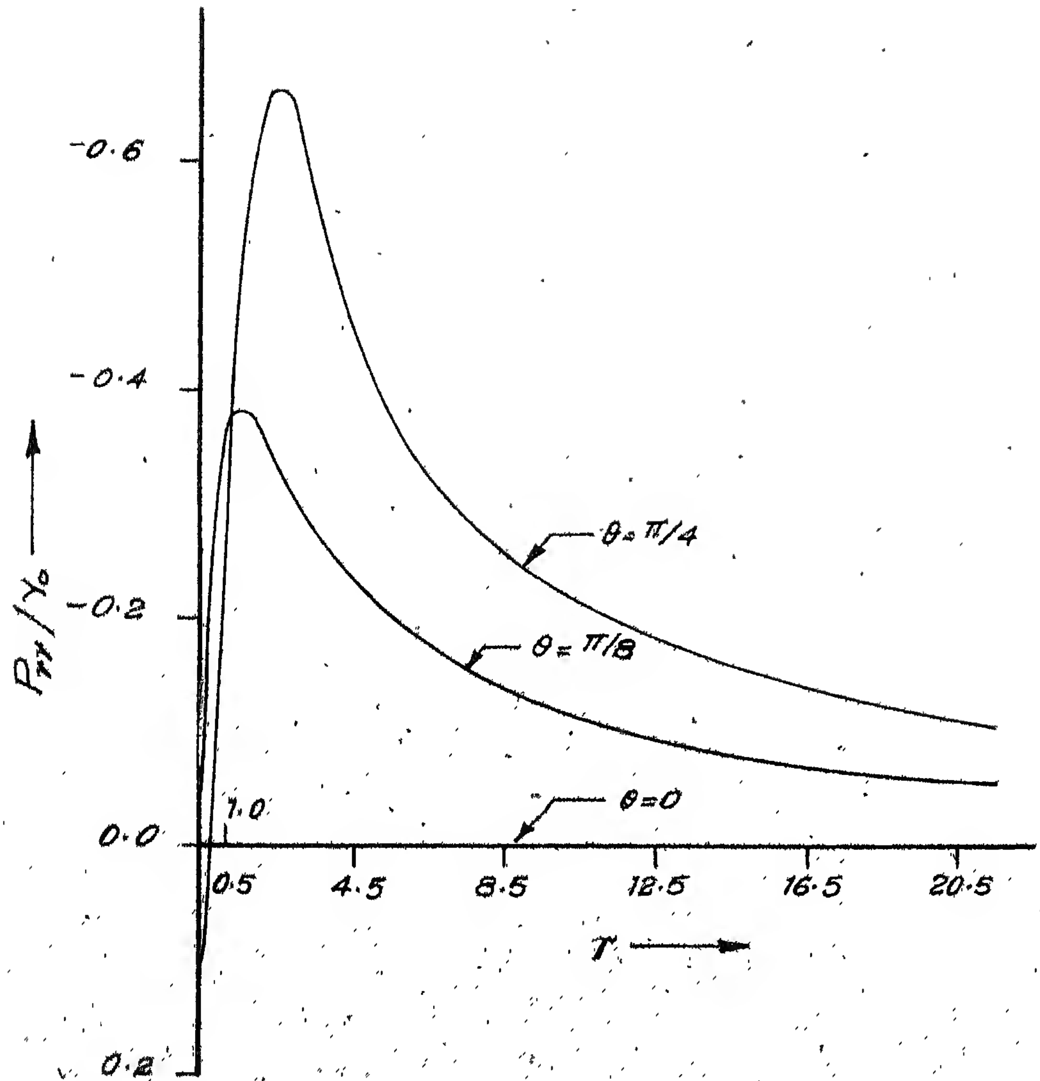


FIG.-2- P_{rr}/Y_0 VS r FOR SOME FIXED θ'S FOR A POINT FORCE Y_0 PERPENDICULAR TO THE AXIS OF SYMMETRY IN AN INFINITE WEDGE OF SEMI-ANGLE $\pi/4$. POINT OF APPLICATION OF FORCE IS (1,0).
POISSON RATIO = 0.25
GRAPHS FOR NEGATIVE VALUES OF θ ARE MIRROR IMAGES OF THOSE FOR POSITIVE VALUES.

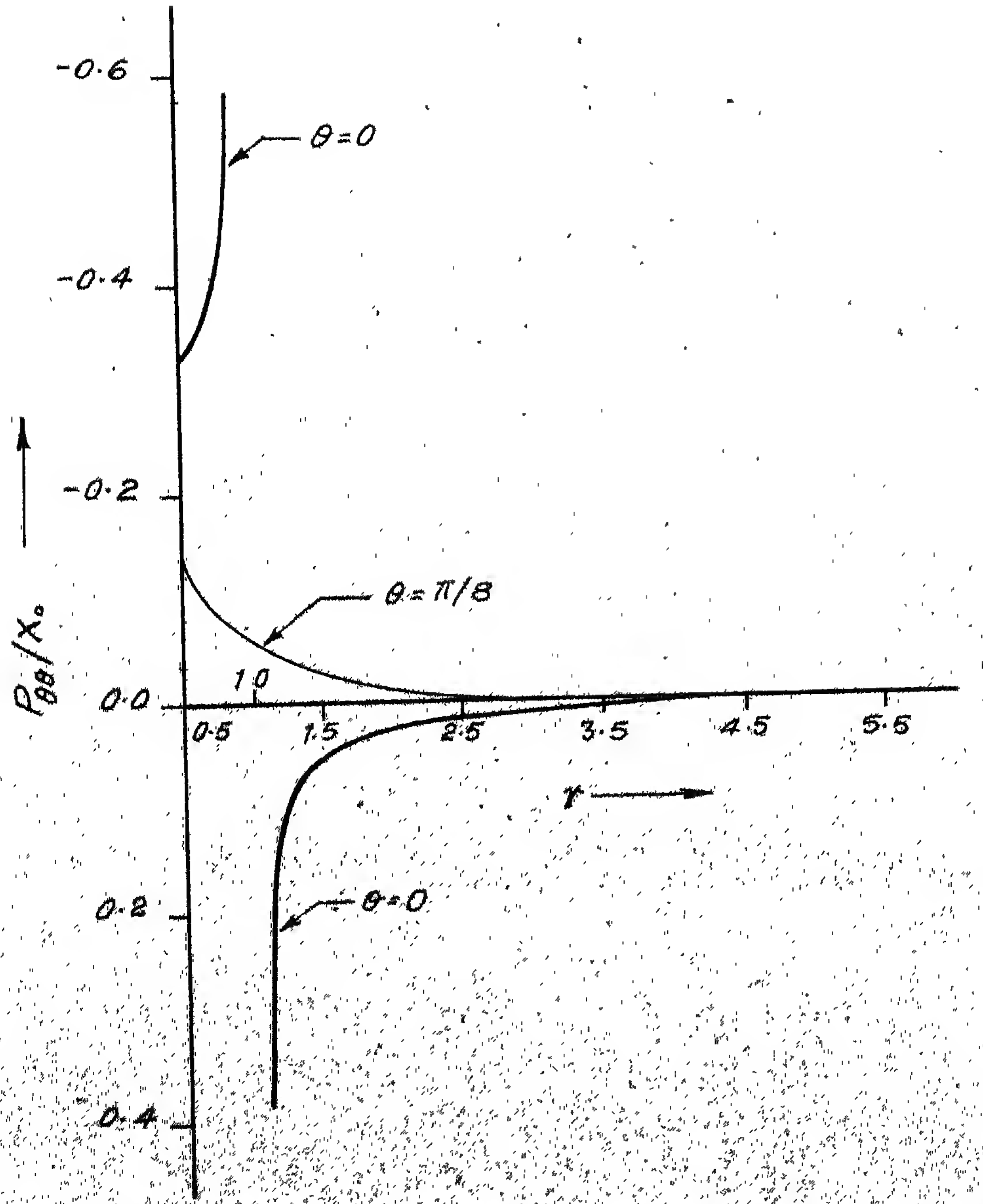


FIG-3- $P_{\theta\theta}/X_0$ VS r FOR SOME FIXED θ'S FOR AN AXIAL FORCE X_0 IN AN INFINITE WEDGE OF SEMI-ANGLE $\pi/4$. POINT OF APPLICATION OF FORCE IS (1,0).
POISSON RATIO = 0.25.

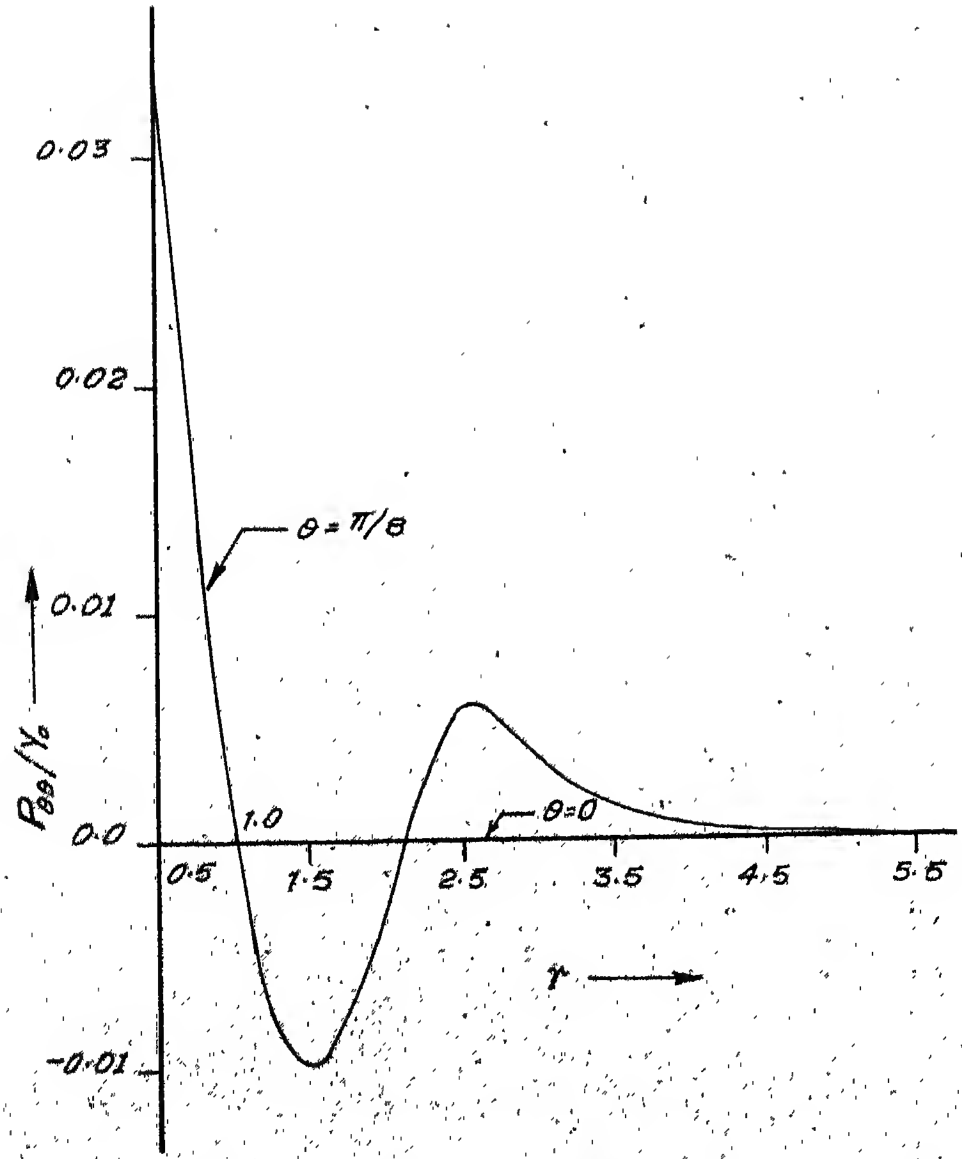


FIG. 4 – $P_{\theta\theta}/Y_0$ VS r FOR SOME FIXED θ'S FOR A POINT FORCE Y_0 PERPENDICULAR TO THE AXIS OF SYMMETRY IN AN INFINITE WEDGE OF SEMI-ANGLE $\pi/4$. POINT OF APPLICATION OF FORCE IS (1,0).
POISSON RATIO = 0.25.
GRAPHS FOR NEGATIVE VALUES OF θ ARE MIRROR IMAGES OF THOSE FOR POSITIVE VALUES.

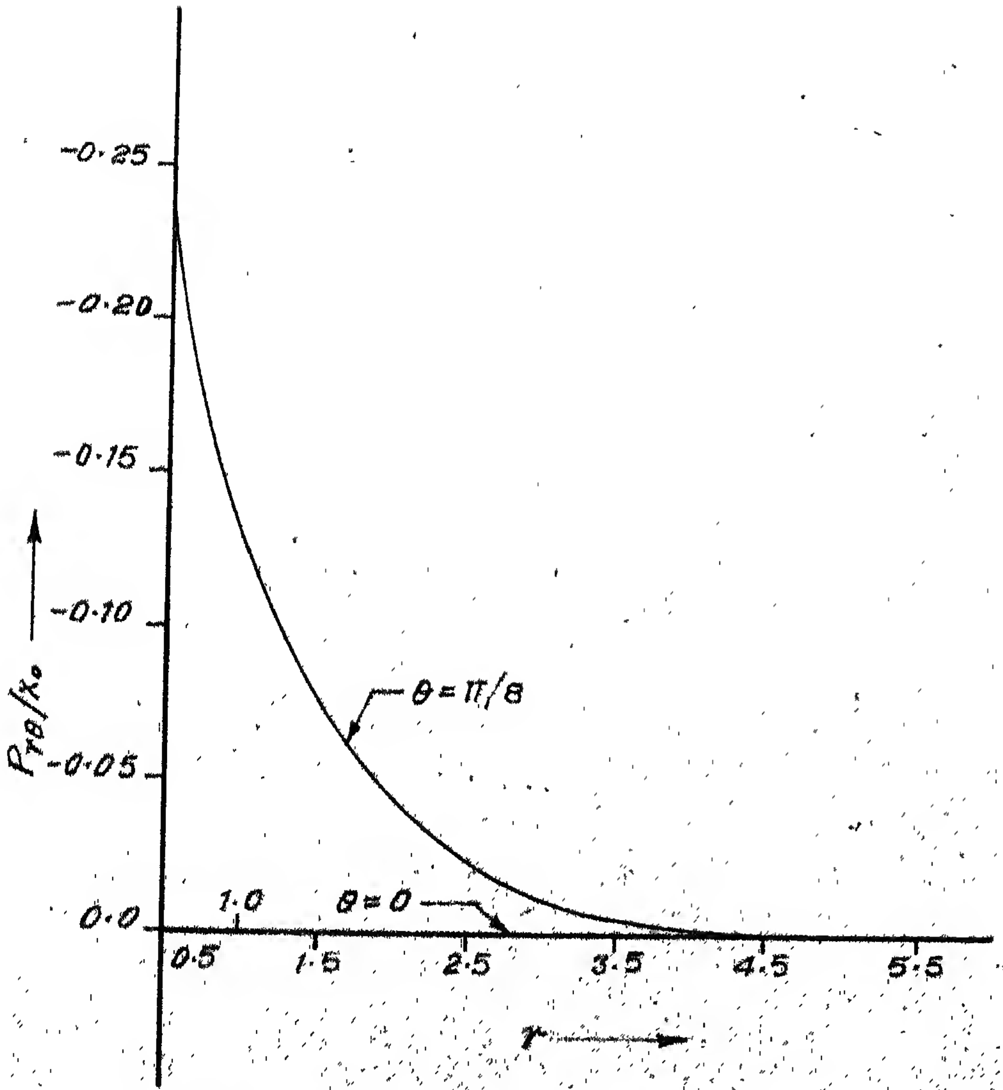


FIG.-5- $P_{r\theta}/X_0$ VS r FOR SOME FIXED θ'S FOR AN AXIAL FORCE X_0 IN AN INFINITE WEDGE OF SEMI-ANGLE $\pi/4$. POINT OF APPLICATION OF FORCE IS (1,0). POISSON RATIO = 0.25.

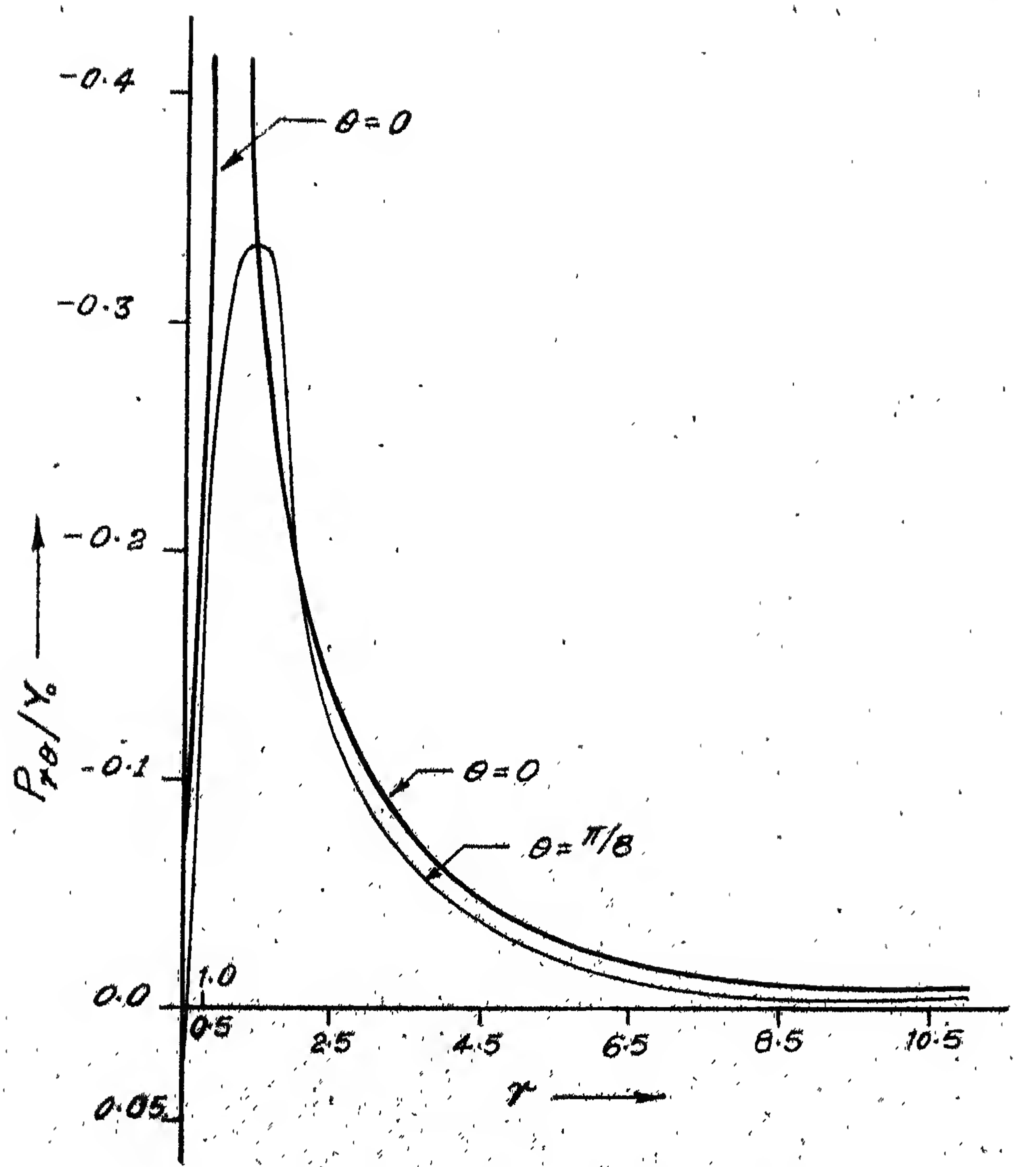


FIG. 6. $P_{r\theta}/Y_0$ VS r FOR SOME FIXED θ'S FOR A POINT FORCE Y_0 PERPENDICULAR TO THE AXIS OF SYMMETRY IN AN INFINITE WEDGE OF SEMIANGLE $\pi/4$. POINT OF APPLICATION OF FORCE IS (1, 0).
POISSON RATIO = 0.25
GRAPHS FOR NEGATIVE VALUES OF θ ARE MIRROR IMAGES OF THOSE FOR POSITIVE VALUES.

APPENDIX II TO CHAPTER X

The quantities F_{11}, F_{22}, F_{33}, F_{44}, F_{55}, F_{66} which appear in the integrals in (208) are given below.

$$F_{11} = \frac{D_{11}}{(D_{11}^2 + D_{22}^2)} \{C_{11} G_{11} - C_{12}G_{12} - C_{21}G_{21} + C_{22}G_{22}\}$$

$$+ \frac{D_{22}}{(D_{11}^2 + D_{22}^2)} \{C_{12}G_{11} + C_{11}G_{12} - C_{22}G_{21} - C_{21}G_{22}\}$$

$$+ \frac{D_{33}}{(D_{33}^2 + D_{44}^2)} \{C'_{11} G_{31} - C'_{12}G_{32} - C'_{21}G_{41} + C'_{22}G_{42}\}$$

$$+ \frac{D_{44}}{(D_{33}^2 + D_{44}^2)} \{C'_{12} G_{31} + C'_{11}G_{32} - C'_{22}G_{41} - C'_{21}G_{42}\},$$

$$F_{22} = - \frac{D_{22}}{(D_{11}^2 + D_{22}^2)} \{C_{11}G_{11} - C_{12}G_{12} - C_{21}G_{21} + C_{22}G_{22}\}$$

$$+ \frac{D_{11}}{(D_{11}^2 + D_{22}^2)} \{C_{12}G_{11} + C_{11}G_{12} - C_{22}G_{21} - C_{21}G_{22}\}$$

$$+ \frac{D_{33}}{(D_{33}^2 + D_{44}^2)} \{C'_{12} G_{31} + C'_{11}G_{32} - C'_{22}G_{41} - C'_{21}G_{42}\} -$$

$$- \frac{D_{44}}{(D_{33}^2+D_{44}^2)} \left\{ C'_{11}G_{31} - C'_{12}G_{32} - C'_{21}G_{41} + C'_{22}G_{42} \right\} ,$$

$$F_{33} = \frac{D_{11}}{(D_{11}^2+D_{22}^2)} \left\{ - \frac{3}{2} (C_{11}G_{51} - C_{12}G_{52}) - \alpha(C_{12}G_{51} + C_{11}G_{52}) \right.$$

$$\left. - C_{21}G_{61} + C_{22}G_{62} \right\} + \frac{D_{22}}{(D_{11}^2+D_{22}^2)} \left\{ \alpha(C_{11}G_{51} - C_{12}G_{52}) \right.$$

$$\left. - \frac{3}{2} (C_{12}G_{51} + C_{11}G_{52}) - C_{22}G_{61} - C_{21}G_{62} \right\}$$

$$+ \frac{D_{33}}{(D_{33}^2+D_{44}^2)} \left\{ - \frac{3}{2} (C'_{11}G_{71} - C'_{12}G_{72}) - \alpha(C'_{12}G_{71} + C'_{11}G_{72}) \right.$$

$$\left. - C'_{21}G_{81} + C'_{22}G_{82} \right\} + \frac{D_{44}}{(D_{33}^2+D_{44}^2)} \left\{ \alpha(C'_{11}G_{71} - C'_{12}G_{72}) \right.$$

$$\left. - \frac{3}{2} (C'_{11}G_{72} + C'_{12}G_{71}) - C'_{21}G_{82} - C'_{22}G_{81} \right\} ,$$

$$F_{44} = \frac{D_{11}}{(D_{11}^2+D_{22}^2)} \left\{ \alpha(C_{11}G_{51} - C_{12}G_{52}) - \frac{3}{2} (C_{12}G_{51} + C_{11}G_{52}) \right.$$

$$\left. - C_{22}G_{61} - C_{21}G_{62} \right\} - \frac{D_{22}}{(D_{11}^2+D_{22}^2)} \left\{ - \frac{3}{2} (C_{11}G_{51} - C_{12}G_{52}) \right.$$

$$\left. - \alpha(C_{12}G_{51} + C_{11}G_{52}) - C_{21}G_{61} + C_{22}G_{62} \right\}$$

$$+ \frac{D_{33}}{(D_{33}^2+D_{44}^2)} \left\{ \alpha(C'_{11}G_{71} - C'_{12}G_{72}) - \frac{3}{2}(C'_{11}G_{72} + C'_{12}G_{71}) - \right.$$

$$-C'_{21}G_{82}-C'_{22}G_{81}\Big\} - \frac{D_{44}}{(D_{33}^2+D_{44}^2)}\Big\{-\tfrac{3}{2}(C'_{11}G_{71}-C'_{12}G_{72})$$

$$-\alpha(C'_{12}G_{71}+C'_{11}G_{72})-C'_{21}G_{81}+C'_{22}G_{82}\Big\} ,$$

$$F_{55} = \frac{D_{11}}{(D_{11}^2+D_{22}^2)}\Big\{C_{11}G_{91}-C_{12}G_{92}-\tfrac{1}{2}(C_{21}G_{71}-C_{22}G_{72})$$

$$+\alpha(C_{22}G_{71}+C_{21}G_{72})\Big\} + \frac{D_{22}}{(D_{11}^2+D_{22}^2)}\Big\{C_{12}G_{91}+C_{11}G_{92}$$

$$-\tfrac{1}{2}(C_{22}G_{71}+C_{21}G_{72})-\alpha(C_{21}G_{71}-C_{22}G_{72})\Big\}$$

$$+\frac{D_{33}}{(D_{33}^2+D_{44}^2)}\Big\{C'_{11}G_{101}-C'_{12}G_{102}+\tfrac{1}{2}(C'_{21}G_{51}-C'_{22}G_{52})$$

$$-\alpha(C'_{22}G_{51}+C'_{21}G_{52})\Big\} + \frac{D_{44}}{(D_{33}^2+D_{44}^2)}\Big\{C'_{12}G_{101}+C'_{11}G_{102}$$

$$+\tfrac{1}{2}(C'_{22}G_{51}+C'_{21}G_{52})+\alpha(C'_{21}G_{51}-C'_{22}G_{52})\Big\} ,$$

$$F_{66} = \frac{D_{11}}{(D_{11}^2+D_{22}^2)}\Big\{C_{12}G_{91}+C_{11}G_{92}-\tfrac{1}{2}(C_{21}G_{72}+C_{22}G_{71})$$

$$-\alpha(C_{21}G_{71}-C_{22}G_{72})\Big\} - \frac{D_{22}}{(D_{11}^2+D_{22}^2)}\Big\{C_{11}G_{91}-C_{12}G_{92}-$$

$$- \frac{1}{2}(C_{21}G_{71} - C_{22}G_{72}) + \alpha(C_{22}G_{71} + C_{21}G_{72})\Big\}$$

$$+ \frac{D_{33}}{(D_{33}^2+D_{44}^2)} \Big\{ C'_{12}G_{101} + C'_{11}G_{102} + \frac{1}{2}(C'_{22}G_{51} + C'_{21}G_{52})$$

$$+ \alpha(C'_{21}G_{51} - C'_{22}G_{52})\Big\} - \frac{D_{44}}{(D_{33}^2+D_{44}^2)} \Big\{ C'_{11}G_{101} - C'_{12}G_{102}$$

$$+ \frac{1}{2}(C'_{21}G_{51} - C'_{22}G_{52}) - \alpha(C'_{22}G_{51} + C'_{21}G_{52})\Big\} \quad ,$$

where

$$G_{11} = - \frac{3}{2}\sin(\theta+\theta_o)\sin\left(\frac{\theta_o-\theta}{2}\right)\cosh\alpha(\theta_o-\theta)$$

$$+ \frac{3}{2}\sin\left(\frac{\theta+\theta_o}{2}\right)\sin(\theta_o-\theta)\cosh\alpha(\theta_o+\theta)$$

$$-2\cos\left(\frac{\theta+3\theta_o}{2}\right)\cosh\alpha(\theta_o-\theta) + 2\cos\left(\frac{\theta-3\theta_o}{2}\right)\cosh\alpha(\theta+\theta_o)$$

$$+ \alpha\sin(\theta+\theta_o)\cos\left(\frac{\theta_o-\theta}{2}\right)\sinh\alpha(\theta_o-\theta)$$

$$- \alpha\cos\left(\frac{\theta+\theta_o}{2}\right)\sin(\theta_o-\theta)\sinh\alpha(\theta+\theta_o) \quad ,$$

$$G_{12} = \alpha\sin(\theta+\theta_o)\sin\left(\frac{\theta_o-\theta}{2}\right)\cosh\alpha(\theta_o-\theta) - \alpha\sin\left(\frac{\theta+\theta_o}{2}\right)\sin(\theta_o-\theta)\cdot$$

$$\cosh\alpha(\theta+\theta_o) + \frac{3}{2}\sin(\theta+\theta_o)\cos\left(\frac{\theta_o-\theta}{2}\right)\sinh\alpha(\theta_o-\theta)$$

$$- \frac{3}{2}\cos\left(\frac{\theta+\theta_o}{2}\right)\sin(\theta_o-\theta)\sinh\alpha(\theta+\theta_o) -$$

$$- 2\sin\left(\frac{\theta+3\theta_o}{2}\right)\sinh\alpha(\theta_o-\theta) - 2\sin\left(\frac{\theta-3\theta_o}{2}\right)\sinh\alpha(\theta+\theta_o),$$

$$G_{21} = \frac{1}{2}\cos\left(\frac{\theta+\theta_o}{2}\right)\sin(\theta_o-\theta)\cosh\alpha(\theta+\theta_o) - \frac{1}{2}\sin(\theta+\theta_o)\cdot$$

$$\cos\left(\frac{\theta_o-\theta}{2}\right)\cosh\alpha(\theta_o-\theta) - \sin\left(\frac{\theta-3\theta_o}{2}\right)\cosh\alpha(\theta+\theta_o)$$

$$- \sin\left(\frac{\theta+3\theta_o}{2}\right)\cosh\alpha(\theta_o-\theta) - \alpha\sin\left(\frac{\theta+\theta_o}{2}\right)\sin(\theta_o-\theta)\cdot$$

$$\sinh\alpha(\theta+\theta_o) + \alpha\sin(\theta+\theta_o)\sin\left(\frac{\theta_o-\theta}{2}\right)\sinh\alpha(\theta_o-\theta),$$

$$G_{22} = \cos\left(\frac{\theta+3\theta_o}{2}\right)\sinh\alpha(\theta_o-\theta) - \cos\left(\frac{\theta-3\theta_o}{2}\right)\sinh\alpha(\theta+\theta_o)$$

$$+ \frac{1}{2}\sin\left(\frac{\theta+\theta_o}{2}\right)\sin(\theta_o-\theta)\sinh\alpha(\theta+\theta_o) - \frac{1}{2}\sin(\theta+\theta_o)\cdot$$

$$\sin\left(\frac{\theta_o-\theta}{2}\right)\sinh\alpha(\theta_o-\theta) + \alpha\cos\left(\frac{\theta+\theta_o}{2}\right)\sin(\theta_o-\theta)\cdot$$

$$\cosh\alpha(\theta+\theta_o) - \alpha\sin(\theta+\theta_o)\cos\left(\frac{\theta_o-\theta}{2}\right)\cosh\alpha(\theta_o-\theta),$$

$$G_{31} = -\frac{3}{2}\sin\left(\frac{\theta+\theta_o}{2}\right)\sin(\theta_o-\theta)\cosh\alpha(\theta_o+\theta)$$

$$- \frac{3}{2}\sin(\theta+\theta_o)\sin\left(\frac{\theta_o-\theta}{2}\right)\cosh\alpha(\theta_o-\theta) - 2\cos\left(\frac{\theta-3\theta_o}{2}\right)\cdot$$

$$\cdot\cosh\alpha(\theta+\theta_o) - 2\cos\left(\frac{\theta+3\theta_o}{2}\right)\cosh\alpha(\theta_o-\theta)$$

$$+ \alpha\cos\left(\frac{\theta+\theta_o}{2}\right)\sin(\theta_o-\theta)\sinh\alpha(\theta+\theta_o) + \alpha\sin(\theta+\theta_o)\cos\left(\frac{\theta_o-\theta}{2}\right)\cdot$$

$$\sinh\alpha(\theta_o-\theta),$$

$$G_{32} = \frac{3}{2}\cos\left(\frac{\theta+\theta_o}{2}\right)\sin(\theta_o-\theta)\sinh\alpha(\theta+\theta_o)$$

$$+ \frac{3}{2}\sin(\theta+\theta_o)\cos\left(\frac{\theta_o-\theta}{2}\right)\sinh\alpha(\theta_o-\theta)$$

$$+ 2\sin\left(\frac{\theta-3\theta_o}{2}\right)\sinh\alpha(\theta+\theta_o) - 2\sin\left(\frac{\theta+3\theta_o}{2}\right)\cdot$$

$$\sinh\alpha(\theta_o-\theta) + \alpha\sin\left(\frac{\theta+\theta_o}{2}\right)\sin(\theta_o-\theta)\cosh\alpha(\theta+\theta_o)$$

$$+ \alpha\sin(\theta+\theta_o)\sin\left(\frac{\theta_o-\theta}{2}\right)\cosh\alpha(\theta_o-\theta)\ ,$$

$$G_{41} = \frac{1}{2}\cos\left(\frac{\theta+\theta_o}{2}\right)\sin(\theta_o-\theta)\cosh\alpha(\theta+\theta_o) + \frac{1}{2}\sin(\theta_o+\theta)\cdot$$

$$\cdot\cos\left(\frac{\theta_o-\theta}{2}\right)\cosh\alpha(\theta_o-\theta) - \alpha\sin\left(\frac{\theta+\theta_o}{2}\right)\sin(\theta_o-\theta)\cdot$$

$$\sinh\alpha(\theta+\theta_o) - \alpha\sin(\theta+\theta_o)\sin\left(\frac{\theta_o-\theta}{2}\right)\sinh\alpha(\theta_o-\theta)$$

$$- \sin\left(\frac{\theta-3\theta_o}{2}\right)\cosh\alpha(\theta+\theta_o) + \sin\left(\frac{\theta+3\theta_o}{2}\right)\cosh\alpha(\theta_o-\theta),$$

$$G_{42} = -\cos\left(\frac{\theta-3\theta_o}{2}\right)\sinh\alpha(\theta+\theta_o) - \cos\left(\frac{\theta+3\theta_o}{2}\right)\sinh\alpha(\theta_o-\theta)$$

$$+ \frac{1}{2}\sin\left(\frac{\theta_o+\theta}{2}\right)\sin(\theta_o-\theta)\sinh\alpha(\theta+\theta_o)$$

$$+ \frac{1}{2}\sin(\theta+\theta_o)\sin\left(\frac{\theta_o-\theta}{2}\right)\sinh\alpha(\theta_o-\theta) +$$

$$+ \alpha \cos\left(\frac{\theta+\theta_o}{2}\right) \sin(\theta_o - \theta) \cosh \alpha(\theta+\theta_o)$$

$$+ \alpha \sin(\theta + \theta_o) \cos\left(\frac{\theta_o-\theta}{2}\right) \cosh \alpha(\theta_o-\theta) \quad ,$$

$$G_{51} = \sin(\theta_o+\theta) \sin\left(\frac{\theta_o-\theta}{2}\right) \cosh \alpha(\theta_o-\theta)$$

$$- \cosh \alpha(\theta_o+\theta) \sin\left(\frac{\theta+\theta_o}{2}\right) \sin(\theta_o-\theta),$$

$$G_{52} = \cos\left(\frac{\theta+\theta_o}{2}\right) \sin(\theta_o-\theta) \sinh \alpha(\theta+\theta_o)$$

$$- \sinh \alpha(\theta_o-\theta) \sin(\theta+\theta_o) \cos\left(\frac{\theta_o-\theta}{2}\right),$$

$$G_{61} = \frac{1}{2} \cos\left(\frac{\theta+\theta_o}{2}\right) \sin(\theta_o-\theta) \cosh\alpha(\theta+\theta_o) - \frac{1}{2} \sin(\theta+\theta_o) \cos\left(\frac{\theta_o-\theta}{2}\right) \cdot$$

$$\cdot \cosh \alpha(\theta_o-\theta) + \sin\left(\frac{\theta-3\theta_o}{2}\right) \cosh \alpha(\theta+\theta_o) + \sin\left(\frac{\theta+3\theta_o}{2}\right) \cdot$$

$$\cdot \cosh \alpha(\theta_o-\theta) - \alpha \sin(\theta_o+\theta) \sin\left(\frac{\theta_o-\theta}{2}\right) \sinh \alpha(\theta-\theta_o)$$

$$+ \alpha \sin\frac{(\theta+\theta_o)}{2} \sin(\theta-\theta_o) \cdot \sinh \alpha(\theta+\theta_o) \quad ,$$

$$G_{62} = \frac{1}{2} \sin(\theta+\theta_o) \sin\left(\frac{\theta_o-\theta}{2}\right) \sinh \alpha(\theta-\theta_o) - \frac{1}{2} \sin\left(\frac{\theta+\theta_o}{2}\right) \cdot$$

$$\cdot \sin(\theta-\theta_o) \sinh \alpha(\theta+\theta_o) + \alpha \cos\left(\frac{\theta+\theta_o}{2}\right) \sin(\theta_o-\theta) \cdot$$

$$\cdot \cosh \alpha(\theta+\theta_o) - \alpha \sin(\theta+\theta_o) \cos\left(\frac{\theta_o-\theta}{2}\right) \cosh \alpha(\theta-\theta_o)$$

$$+ \cos\left(\frac{\theta-3\theta_o}{2}\right) \sinh \alpha(\theta+\theta_o) + \cos\left(\frac{\theta+3\theta_o}{2}\right) \sinh \alpha(\theta-\theta_o) \quad ,$$

$$G_{71} = -\sin\left(\frac{\theta+\theta_0}{2}\right)\sin(\theta-\theta_0)\cosh\alpha(\theta+\theta_0)$$

$$+\sin(\theta+\theta_0)\sin\left(\frac{\theta_0-\theta}{2}\right)\cosh\alpha(\theta_0-\theta),$$

$$G_{72} = \cos\left(\frac{\theta+\theta_0}{2}\right)\sin(\theta-\theta_0)\sinh\alpha(\theta+\theta_0)$$

$$-\sinh\alpha(\theta_0-\theta)\sin(\theta_0+\theta)\cos\left(\frac{\theta_0-\theta}{2}\right),$$

$$G_{81} = \frac{1}{2}\cos\left(\frac{\theta+\theta_0}{2}\right)\sin(\theta_0-\theta)\cosh\alpha(\theta_0+\theta) + \frac{1}{2}\sin(\theta+\theta_0)\cdot$$

$$\cos\left(\frac{\theta-\theta_0}{2}\right)\cosh\alpha(\theta_0-\theta) + \alpha\sin\left(\frac{\theta+\theta_0}{2}\right)\sin(\theta-\theta_0)\cdot$$

$$\cdot\sinh\alpha(\theta+\theta_0) - \alpha\sin(\theta+\theta_0)\sin\left(\frac{\theta_0-\theta}{2}\right)\sinh\alpha(\theta_0-\theta)$$

$$+\sin\left(\frac{\theta-3\theta_0}{2}\right)\cosh\alpha(\theta_0+\theta) - \sin\left(\frac{\theta+3\theta_0}{2}\right)\cosh\alpha(\theta_0-\theta),$$

$$G_{82} = \cos\left(\frac{\theta-3\theta_0}{2}\right)\sinh\alpha(\theta+\theta_0) + \cos\left(\frac{\theta+3\theta_0}{2}\right)\sinh\alpha(\theta_0-\theta)$$

$$-\frac{1}{2}\sin\left(\frac{\theta+\theta_0}{2}\right)\sin(\theta-\theta_0)\sinh\alpha(\theta+\theta_0) + \frac{1}{2}\sin(\theta_0+\theta)\cdot$$

$$\cdot\sin\left(\frac{\theta_0-\theta}{2}\right)\sinh\alpha(\theta_0-\theta) + \alpha\cos\left(\frac{\theta_0+\theta}{2}\right)\sin(\theta_0-\theta)\cdot$$

$$\cdot\cosh\alpha(\theta+\theta_0) + \alpha\sin(\theta+\theta_0)\cos\left(\frac{\theta-\theta_0}{2}\right)\cosh\alpha(\theta_0-\theta),$$

$$G_{91} = \frac{1}{2}\cos\left(\frac{\theta+\theta_0}{2}\right)\sin(\theta_0-\theta)\cosh\alpha(\theta+\theta_0)$$

$$+ \ \frac{1}{2}\sin(\theta+\theta_o)\ \cos(\frac{\theta-\theta_o}{2})\ \cosh\alpha(\theta_o-\theta)$$

$$- \ \alpha\sin(\frac{\theta+\theta_o}{2})\ \sin(\theta_o-\theta)\ \sinh\alpha(\theta+\theta_o)$$

$$- \ \alpha\sin(\theta+\theta_o)\ \sin(\frac{\theta_o-\theta}{2})\ \sinh\alpha(\theta_o-\theta)$$

$$- \ \sin(\frac{\theta_o-3\theta}{2})\ \cosh\alpha(\theta_o+\theta) - \sin(\frac{\theta_o+3\theta}{2})\cosh\alpha(\theta_o-\theta),$$

$$G_{92} = \alpha\cos(\frac{\theta+\theta_o}{2})\sin(\theta_o-\theta)\ \cosh\alpha(\theta+\theta_o) + \alpha\sin(\theta+\theta_o)\cos(\frac{\theta-\theta_o}{2})\cdot$$

$$\cosh\alpha(\theta_o-\theta) - \frac{1}{2}\sin(\frac{\theta_o+\theta}{2})\ \sin\ (\theta-\theta_o)\sinh\alpha(\theta+\theta_o)$$

$$+ \ \frac{1}{2}\sin(\theta+\theta_o)\ \sin(\frac{\theta_o-\theta}{2})\ \sinh\alpha(\theta_o-\theta) - \cos(\frac{\theta_o-3\theta}{2})\cdot$$

$$\sinh\alpha(\theta+\theta_o) - \cos(\frac{\theta_o+3\theta}{2})\ \sinh\alpha(\theta_o-\theta)\ ,$$

$$G_{101} = \frac{1}{2}\cos(\frac{\theta+\theta_o}{2})\ \sin(\theta-\theta_o)\ \cosh\alpha(\theta+\theta_o) + \frac{1}{2}\sin(\theta+\theta_o)\cdot$$

$$\cos(\frac{\theta_o-\theta}{2})\ \cosh\alpha(\theta-\theta_o) + \alpha\sin(\theta+\theta_o)\ \sin(\frac{\theta_o-\theta}{2})\cdot$$

$$\sinh\alpha(\theta-\theta_o) + \alpha\sin(\frac{\theta+\theta_o}{2})\ \sin(\theta_o-\theta)\ \sinh\alpha(\theta+\theta_o)$$

$$+ \sin(\frac{\theta_o-3\theta}{2})\ \cosh\alpha(\theta+\theta_o) - \sin(\frac{\theta_o+3\theta}{2})\cosh\alpha(\theta-\theta_o),$$

$$G_{102} = \alpha \cos(\frac{\theta+\theta_o}{2}) \sin(\theta-\theta_o)\cosh \alpha(\theta+\theta_o) + \alpha \sin(\theta+\theta_o)\cdot$$

$$\cos(\frac{\theta_o-\theta}{2}) \cosh \alpha(\theta-\theta_o) + \frac{1}{2} \sin(\theta_o+ \theta)\sin(\frac{\theta-\theta_o}{2})\cdot$$

$$\sinh \alpha(\theta-\theta_o) - \frac{1}{2} \sin(\frac{\theta+\theta_o}{2}) \sin(\theta_o-\theta) \sinh \alpha(\theta+\theta_o)$$

$$+ \cos(\frac{\theta_o-3\theta}{2}) \sinh \alpha(\theta+\theta_o) + \cos(\frac{\theta_o+3\theta}{2}) \sinh \alpha(\theta-\theta_o),$$

$$D_{11} = 2\{- \frac{1}{2} \sin 2\theta_o+ \sin \theta_o \cosh 2 \alpha \theta_o\},$$

$$D_{22} = -2(\cos \theta_o \sinh 2\alpha \theta_o - \alpha \sin 2 \theta_o),$$

$$D_{33} = -2(\frac{1}{2} \sin 2 \theta_o+ \sin \theta_o\cosh 2\alpha \theta_o) ,$$

$$D_{44} = 2\{\alpha \sin 2 \theta_o + \cos \theta_o \sinh 2 \alpha \theta_o\}.$$

The quantities C_{11}, C_{12}, C'_{11}, C'_{12} etc. occurring in the expressions of F_{11}, F_{22} etc. are given below.

$$C_{11} = \frac{1}{2} (C_1+ \bar{C}_1),$$

$$C_{12} = - \frac{i}{2} (C_1 - \bar{C}_1) ,$$

$$C'_{11} = \frac{1}{2} (C'_1 + \bar{C}'_1) ,$$

$$C'_{12} = - \frac{i}{2} (C'_1 - \bar{C}'_1),$$

$$C_{21} = \frac{1}{2}(C_2 + \overline{C}_2) ,$$

$$C_{22} = -\frac{i}{2}(C_2 - \overline{C}_2),$$

$$C'_{21} = \frac{1}{2}(C'_2 + \overline{C}'_2) ,$$

$$C'_{22} = -\frac{i}{2}(C'_2 - \overline{C}'_2) ,$$

$$C_1 = f_{11}(\alpha, \theta_o) + f_{11}(\alpha, -\theta_o)$$

$$C'_1 = -(f_{11}(\alpha, \theta_o) + f_{11}(\alpha, -\theta_o)),$$

$$C_2 = f_{22}(\alpha, \theta_o) - f_{22}(\alpha, -\theta_o) ,$$

$$C'_2 = f_{22}(\alpha, \theta_o) + f_{22}(\alpha, -\theta_o) ,$$

$$f_{11}(\alpha, \theta_o) = \frac{X_o}{4(1-\nu)\sin^3\theta_1 \cosh \pi\alpha}\Big[(1-2\nu)\sin^2\theta_1\Big\{$$

$$-r_o^3 \sin\theta_o \cos\frac{\theta_1}{2}\cosh\alpha(\pi-\theta_1)$$

$$+r_o^2(h\sin 2\theta_o - k\cos 2\theta_o)\cos\frac{\theta_1}{2}\cosh\alpha(\pi-\theta_1)\Big\}$$

$$+\cos\theta_o(h\sin\theta_o - k\cos\theta_o)A_{11} - H_{11}A_{22} - H_{22}A_{33}\Big]$$

$$+\frac{i X_o}{4(1-\nu)\sin^3\theta_1 \cosh\pi\alpha}\Big[(1-2\nu)\sin^2\theta_1\Big\{$$

$$-r_o^3\sin\theta_o\sin\frac{\theta_1}{2}\sinh\alpha(\pi-\theta_1) -$$

$$- r_o^2(h \sin 2\theta_o - k \cos 2\theta_o) \sin \frac{\theta 1}{2} \sinh \alpha(\pi-\theta_1)\Big\}$$

$$+ \cos \theta_o(h \sin \theta_o - k \cos\theta_o) \; A_{44} - H_{11}A_{55} - H_{22}A_{66}\Big]$$

$$+ \frac{Y_o}{4(1-\nu)\sin^3\theta_1 \cosh \pi \alpha}\Big[(1-2\nu)\sin^2\theta_1 \Big\{$$

$$r_o^3 \cos \theta_o \cos \frac{\theta_1}{2} \cosh \alpha(\pi-\theta_1)$$

$$- r_o^2 (h \cos 2\theta_o + k \sin 2\theta_o) \cos \frac{\theta_1}{2} \cosh \alpha(\pi - \theta_1)\Big\}$$

$$+ \sin \theta_o(h \sin \theta_o - k \cos \theta_o) \; A_{11} - H_{33}A_{22} - H_{44}A_{33}\Big]$$

$$+ \frac{i\, Y_o}{4(1-\nu)\sin^3\theta_1 \cosh \pi\alpha}\Big[(1-2\nu)\sin^2\theta_1 \Big\{ r_o^3 \cos \theta_o \cdot$$

$$\sin \frac{\theta_1}{2} \sinh \alpha(\pi- \theta_1) + r_o^2 (h \cos 2\theta_o + k \sin 2\theta_o)$$

$$\cdot \sin \frac{\theta_1}{2} \sinh \alpha(\pi - \theta_1)\Big\} + \sin \theta_o(h \sin \theta_o - k\cos\theta_o) \cdot$$

$$\cdot A_{44} - H_{33}A_{55} - H_{44}A_{66}\Big] ,$$

where $\cos \theta_1 = \dfrac{h \cos \theta_o + k \sin \theta_o}{r_o}$,

$$r_o^2 = h^2 + k^2 ,$$

$$A_{11} = r_o^2 \Big\{ \frac{3}{2} \sin \frac{\theta_1}{2} \sin \theta_1 \cosh \alpha(\pi-\theta_1) + \alpha \sin \theta_1 \cos \frac{\theta_1}{2} \cdot$$

$$\sinh \alpha(\pi- \theta_1) + \cos \frac{3\theta_1}{2} \cosh \alpha(\pi - \theta_1)\Big\} ,$$

$$A_{22} = r_o\left\{\frac{1}{2}\sin\frac{\theta_1}{2}\sin\theta_1\cosh\alpha(\pi-\theta_1) - \alpha\sin\theta_1\cos\frac{\theta_1}{2}\cdot\right.$$

$$\left.\sinh\alpha(\pi-\theta_1) - \cos\frac{\theta_1}{2}\cosh\alpha(\pi-\theta_1)\right\},$$

$$A_{33} = \frac{1}{2}\sin\frac{\theta_1}{2}\sin\frac{3\theta_1}{2}\cosh\alpha(\pi-\theta_1) + \alpha\sin\theta_1\cos\frac{3\theta_1}{2}\cdot$$

$$\sinh\alpha(\pi-\theta_1) + \cos\frac{\theta_1}{2}\cosh\alpha(\pi-\theta_1),$$

$$A_{44} = r_o^2\left\{\sin\frac{3\theta_1}{2}\sinh\alpha(\pi-\theta_1) + \alpha\sin\theta_1\sin\frac{\theta_1}{2}\cosh\alpha(\pi-\theta_1)\right.$$

$$\left.-\frac{3}{2}\sin\theta_1\cos\frac{\theta_1}{2}\sinh\alpha(\pi-\theta_1)\right\},$$

$$A_{55} = r_o\left\{-\sin\frac{\theta_1}{2}\sinh\alpha(\pi-\theta_1) + \alpha\sin\theta_1\sin\frac{\theta_1}{2}\cdot\right.$$

$$\left.\cosh\alpha(\pi-\theta_1) + \frac{1}{2}\sin\theta_1\cos\frac{\theta_1}{2}\sinh\alpha(\pi-\theta_1)\right\},$$

$$A_{66} = -\sin\frac{\theta_1}{2}\sinh\alpha(\pi-\theta_1) - \alpha\sin\theta_1\sin\frac{3\theta_1}{2}\cosh\alpha(\pi-\theta_1)$$

$$+\frac{1}{2}\sin\theta_1\cos\frac{3\theta_1}{2}\sinh\alpha(\pi-\theta_1),$$

$$H_{11} = 2hk\cos^3\theta_o + \sin\theta_o\cos^2\theta_o(k^2-h^2) - h^2\sin\theta_o,$$

$$H_{22} = h^2k\cos 2\theta_o + h\sin\theta_o\cos\theta_o(k^2-h^2),$$

$$H_{33} = -2hk\sin^3\theta_o + k^2\cos\theta_o + (k^2-h^2)\sin^2\theta_o\cos\theta_o,$$

$$H_{44} = k\left\{(k^2-h^2)\sin\theta_o\cos\theta_o + hk\cos 2\theta_o\right\}.$$

The expression for $f_{11}(\alpha, -\theta_o)$ may be obtained from the expression of $f_{11}(\alpha, \theta_o)$ by replacing θ_o by $-\theta_o$ and θ_1 by θ_2 in $f_{11}(\alpha, \theta_o)$ and so also in A_{11}, A_{22}, A_{33}, A_{44}, A_{55}, A_{66} and H_{11}, H_{22}, H_{33} and H_{44}. θ_2 is given by

$$\cos\theta_2 = \frac{h\cos\theta_o - k\sin\theta_o}{r_o} .$$

$$f_{22}(\alpha, \theta_o) = \frac{X_o}{4(1-\nu)\sin^3\theta_1\cosh\pi\alpha}\Big[(1-2\nu)\sin^2\theta_1\cos\frac{\theta_1}{2} \cdot$$

$$\cosh\alpha(\pi-\theta_1)\left\{-r_o^3\cos\theta_o + r_o^2(h\cos 2\theta_o + k\sin 2\theta_o)\right\}$$

$$+ (k\cos\theta_o - h\sin\theta_o)^2\left\{-A_{22}\cos\theta_o - h A_{33}\right\}\Big]$$

$$+ \frac{i\,X_o}{4(1-\nu)\sin^3\theta_1\cosh\pi\alpha}\Big[(1-2\nu)\sin^2\theta_1\sin\frac{\theta_1}{2} \cdot$$

$$\sinh\alpha(\pi-\theta_1)\left\{-r_o^3\cos\theta_o - r_o^2(h\cos 2\theta_o + k\sin 2\theta_o)\right\}$$

$$+ (k\cos\theta_o - h\sin\theta_o)^2\left\{-A_{55}\cos\theta_o - h A_{66}\right\}\Big]$$

$$+ \frac{Y_o}{4(1-\nu)\sin^3\theta_1\cosh\pi\alpha}\Big[(1-2\nu)\sin^2\theta_1\cos\frac{\theta_1}{2}\cosh\alpha(\pi-\theta_1) \cdot$$

$$\left\{-r_o^3\sin\theta_o + r_o^2(h\sin 2\theta_o - k\cos 2\theta_o)\right\} +$$

$$+ (k \cos \theta_o - h \sin \theta_o)^2 \{- A_{22} \sin \theta_o - k A_{33}\}\Big]$$

$$+ \frac{i\, Y_o}{4(1-\nu)\sin^3\theta_1 \cosh \pi\alpha} \Big[(1-2\nu) \sin \frac{\theta_1}{2} \sinh \alpha(\pi-\theta_1) \cdot$$

$$\{- r_o^3 \sin \theta_o - r_o^2 (h \sin 2\theta_o - k \cos 2\theta_o)\}$$

$$+ (k \cos \theta_o - h \sin \theta_o)^2 \{- A_{55} \sin \theta_o - k A_{66}\}\Big] \quad ,$$

the expression for $f_{22}(\alpha, -\theta_o)$ may be obtained from the expression of $f_{22}(\alpha, \theta_o)$ by replacing θ_o by $-\theta_o$ and θ_1 by θ_2 in $f_{22}(\alpha, \theta_o)$ and so also in A_{11}, A_{22}, A_{33}, A_{44}, A_{55} and A_{66} .

CHAPTER XI

CIRCULAR INCLUSION IN A WEDGE

The method described in chapter X can also be used to solve some other interesting problems. This has been done in this chapter by solving the problem of circular inclusion in a wedge. The problem may be stated as follows:

Consider the infinite elastic wedge described in chapter X. The system is in a state of plain strain. Thus we are considering only a plane cross-section perpendicular to the z axis of an infinite elastic solid, which is bounded by the two planes $\theta = \pm\,\theta_0$ (R, θ, z ($= Z$) being the cylindrical polar coordinates) ; the edge of the wedge is along z axis. The x axis is along the axis of symmetry and the y axis is perpendicular to x axis. We shall henceforth work in (x,y) plane only. The origin is taken at the vertex of the wedge and is named as O .

Let there be a circular hole of radius r' in this infinite wedge (Figure 3, p.139). The Cartesian components of the centre of this circular hole are (x_o, y_o) . Suppose a circular disc of dimensions slightly bigger than those of the hole is embedded in this hole. Due to misfit, stresses would develop everywhere in the wedge. The problem considered is that of determining this elastic field so developed.

The solution has been achieved in the following manner.

Consider a circular inclusion of radius r' in a two-dimensional infinite elastic solid. The centre of circular inclusion is at a point $z_o = x_o + i\ y_o$ with respect to the origin at O. The boundary of inclusion is denoted by L . The inclusion in the absence of the surrounding material undergoes a displacement whose Cartesian components are characterised by $(\epsilon_1 x, \epsilon_1 y)$ with respect to the origin at z_o ; the centre of L .

Let (u^+, v^+) be the displacement components in Cartesian coordinates of the inclusion and (u^-, v^-) be those of the matrix, where + and - superscripts have the same meaning as in chapter I. At the equilibrium interface if the displacement components for inclusion and matrix are denoted by (u_b^+, v_b^+) and (u_b^-, v_b^-), respectively, then

$$u_b^+ - u_b^- = -\epsilon_1 x \quad ,$$

$$v_b^+ - v_b^- = -\epsilon_1 y \ . \tag{210}$$

The sectionally holomorphic functions $\phi_1(z)$ and $\psi_1(z)$ $(z = x + iy = re^{i\theta})$ having the line of discontinuity L may be found out as discussed in chapter I. It may be noted that the sectionally holomorphic functions $\phi_1(z)$ and $\psi_1(z)$ are determined with respect to the origin at z_o. The sectionally holomorphic functions $\phi(z)$ and $\psi(z)$ with respect to the origin at O may be obtained from $\phi_1(z)$ and $\psi_1(z)$ by the translation of origin ((2)) and are given below.

$$\phi(z) = \phi_1(z-z_o) \quad ,$$
$$\psi(z) = \psi_1(z-z_o) - \bar{z}_o \ \phi_1'(z-z_o) \ . \tag{211}$$

Stresses P_{rr}, $P_{\theta\theta}$ and $P_{r\theta}$ in polar coordinates (r,θ) can be easily calculated from (211) using (10). Let these stresses be called the stresses due to the first stress system. They are denoted by $(P_{rr})_1$, $(P_{\theta\theta})_1$ and $(P_{r\theta})_1$, the subscript 1 is added to the stresses to indicate that they are the first system of stresses. $(P_{\theta\theta})_1$ and $(P_{r\theta})_1$ may be calculated at $\theta = \pm\,\theta_o$. Now apply $(P_{\theta\theta})_2$ and $(P_{r\theta})_2$ equal in

magnitude and opposite in sign to $(P_{\theta\theta})_1$ and $(P_{r\theta})_1$ on the faces $\theta = \pm\,\theta_o$ of the wedge. Thus

$$\left\{(P_{\theta\theta})_2\right\}_{\theta=\pm\,\theta_o} = -\left\{(P_{\theta\theta})_1\right\}_{\theta=\pm\,\theta_o}$$

and $$\left\{(P_{r\theta})_2\right\}_{\theta=\pm\,\theta_o} = -\left\{(P_{r\theta})_1\right\}_{\theta=\pm\,\theta_o} .$$

This will develop stresses everywhere in the wedge. These stresses are called the second system of stresses and we distinguish them by adding subscript 2 to them. Superposition of these two systems of stresses inside the wedge would give the solution to the problem when the circular inclusion which was hitherto present in the infinite plane, is situated in an infinite elastic wedge bounded by the lines $\theta = \pm\,\theta_o$.

We now determine the first system of stresses. When a circular inclusion of radius r' whose centre is at a point z_o ($z_o = x_o + iy_o$) is present in a two-dimensional infinite elastic plane, the sectionally holomorphic functions $\phi(z)$ and $\psi(z)$ for inclusion and matrix are respectively

$$\phi_i(z) = -\frac{2\mu\,\epsilon_1}{(K+1)}\,(z-z_o) ,$$

$$\psi_i(z) = + \frac{2\mu\epsilon_1}{(K+1)} \bar{z}_o \qquad (212)$$

and $\qquad \phi_m(z) = 0 \quad ,$

$$\psi_m(z) = - \frac{4\mu\epsilon_1 r'^2}{(K+1)} \frac{1}{(z-z_o)} \quad , \qquad (213)$$

where the subscript i and m refer to inclusion and matrix respectively. It may be noted that $\phi_i(z)$, $\psi_i(z)$, $\phi_m(z)$ and $\psi_m(z)$ are determined with respect to the origin at O.

The stresses in polar coordinates for matrix and inclusion corresponding to $\phi(z)$ and $\psi(z)$ in (213) and (212) are

$$\{(P_{rr})_m\}_1 = - \frac{4\mu\epsilon_1 r'^2}{(K+1)} \frac{\{r^2+r_{11}^2\cos 2(\theta-\beta_{11})-2rr_{11}\cos(\theta-\beta_{11})\}}{\{r^2+r_{11}^2 - 2rr_{11}\cos(\theta-\beta_{11})\}^2} ,$$

$$\{(P_{\theta\theta})_m\}_1 = \frac{4\mu\epsilon_1 r'^2}{(K+1)} \frac{\{r^2+r_{11}^2\cos 2(\theta-\beta_{11})-2rr_{11}\cos(\theta-\beta_{11})\}}{\{r^2+r_{11}^2 - 2rr_{11}\cos(\theta-\beta_{11})\}^2} ,$$

$$\{(P_{r\theta})_m\}_1 = \frac{4\mu\epsilon_1 r'^2}{(K+1)} \frac{\{r_{11}^2 \sin 2(\theta-\beta_{11})-2rr_{11}\sin(\theta-\beta_{11})\}}{\{r^2+r_{11}^2 - 2rr_{11}\cos(\theta-\beta_{11})\}^2}$$

(214)

and

$$\left\{(P_{rr})_i\right\}_1 = -\frac{4\mu\epsilon_1}{(K+1)},$$

$$\left\{(P_{\theta\theta})_i\right\}_1 = -\frac{4\mu\epsilon_1}{(K+1)}, \quad \left\{(P_{r\theta})_i\right\}_1 = 0, \qquad (215)$$

where $r_{11}^2 = x_o^2 + y_o^2$ and $\beta_{11} = \tan^{-1}(y_o/x_o)$.

On the faces of the wedge

$$\left\{(P_{\theta\theta})_2\right\}_{\theta=\theta_o} = -\frac{4\mu\epsilon_1 r'^2}{(K+1)} \frac{\left\{r^2+r_{11}^2\cos 2(\theta_o-\beta_{11})-2rr_{11}\cos(\theta_o-\beta_{11})\right\}}{\left\{r^2+r_{11}^2-2rr_{11}\cos(\theta_o-\beta_{11})\right\}^2}, \qquad (216)$$

$$\left\{(P_{\theta\theta})_2\right\}_{\theta=-\theta_o} = -\frac{4\mu\epsilon_1 r'^2}{(K+1)} \frac{\left\{r^2+r_{11}^2\cos 2(\theta_o+\beta_{11})-2rr_{11}\cos(\theta_o+\beta_{11})\right.}{\left\{r^2+r_{11}^2-2rr_{11}\cos(\theta_o+\beta_{11})\right\}^2} \qquad (217)$$

$$\left\{(P_{r\theta})_2\right\}_{\theta=\theta_o} = -\frac{4\mu\epsilon_1 r'^2}{(K+1)} \frac{\left\{r_{11}^2\sin 2(\theta_o-\beta_{11})-2rr_{11}\sin(\theta_o-\beta_{11})\right\}}{\left\{r^2+r_{11}^2-2rr_{11}\cos(\theta_o-\beta_{11})\right\}^2}, \qquad (218)$$

$$\left\{(P_{r\theta})_2\right\}_{\theta=-\theta_o} = -\frac{4\mu\epsilon_1 r'^2}{(K+1)} \frac{\left\{-r_{11}^2\sin 2(\theta_o+\beta_{11})+2rr_{11}\sin(\theta_o+\beta_{11})\right\}}{\left\{r^2+r_{11}^2-2rr_{11}\cos(\theta_o+\beta_{11})\right\}^2}. \qquad (219)$$

The solution of this auxiliary problem of a wedge with the boundary conditions given in (216)-(219) can be obtained with the help of the method given in chapter X. As mentioned there, we start with the equations of equilibrium in polar coordinates. Their solution in terms of a biharmonic function $\phi(r,\theta)$ is given in details in ((17)). Stresses in terms of ϕ are given by (191). If $\bar{\Phi}$ is defined as in (192), then solution of (190) is given by (193). Also $(\overline{P_{rr}})_2$, $(\overline{P_{\theta\theta}})_2$ and $(\overline{P_{r\theta}})_2$, the Mellin transforms of P_{rr}, $P_{\theta\theta}$ and $P_{r\theta}$ in (191) are given by (194). The constants A, B, C and D in (194) are determined with the help of the boundary conditions (216) - (219). Multiplying the equations (216), (217), (218) and (219) by r^{s-1} and integrating with respect to r from 0 to ∞, we get

$$\begin{aligned}
\left\{(\overline{P_{\theta\theta}})_2\right\}_{\theta=\theta_0} &= r_{11}^{s-2} f_{22}(s,\theta_0) , \\
\left\{(\overline{P_{\theta\theta}})_2\right\}_{\theta=-\theta_0} &= r_{11}^{s-2} f_{22}(s, -\theta_0) , \\
\left\{(\overline{P_{r\theta}})_2\right\}_{\theta=\theta_0} &= r_{11}^{s-2} f_{11}(s, \theta_0) , \\
\left\{(\overline{P_{r\theta}})_2\right\}_{\theta=-\theta_0} &= r_{11}^{s-2} f_{11}(s, -\theta_0),
\end{aligned} \tag{220}$$

where $f_{22}(s,\theta_0)$, $f_{22}(s,-\theta_0)$, $f_{11}(s,\theta_0)$ and $f_{11}(s,-\theta_0)$ in

real and imaginary parts are given in (224) below.

It may be noted that all the integrals on the right hand side of (220) are found to be valid for $0 < \mathrm{Re}\,(s) < 1$.

The four constants A, B, C and D may be determined from (194) and (220). They are given by (199), (200), (201) and (202) where $f_{11}(s,\theta_0)$, $f_{11}(s,-\theta_0)$, $f_{22}(s,\theta_0)$ and $f_{22}(s,-\theta_0)$ are different for this problem. Substituting these values of the constants A, B, C and D in (194) and inverting (194) with the help of the inversion formula given in (127), we get (203). It may be noted that the values of C_1, C_1', C_2 and C_2' are different for this problem as $f_{11}(s,\theta_0)$, $f_{11}(s,-\theta_0)$, $f_{22}(s,\theta_0)$ and $f_{22}(s,-\theta_0)$ are different for this problem. Other quantities remain the same.

The choice of $\gamma = 1/2$ which in turn implies

$$s = \frac{1}{2} + i\alpha$$

is justified by the same arguments as given in chapter X, p. 171. Putting $s = \frac{1}{2} + i\alpha$ in (203) and separating them in real and imaginary parts, we get the second system of stresses $(P_{rr})_2$, $(P_{\theta\theta})_2$ and $(P_{r\theta})_2$ as

$$(P_{rr})_2 = -\frac{1}{\pi r^{1/2} r_{11}^{3/2}} \int_0^\infty (F_{11}\cos m_2\alpha - F_{22}\sin m_2\alpha)\, d\alpha ,$$

$$(P_{\theta\theta})_2 = \frac{1}{\pi r^{1/2} r_{11}^{3/2}} \int_0^\infty (F_{33}\cos m_2\alpha - F_{44}\sin m_2\alpha)d\alpha ,$$

$$(P_{r\theta})_2 = \frac{1}{\pi r^{1/2} r_{11}^{3/2}} \int_0^\infty (F_{55} \cos m_2\alpha - F_{66}\sin m_2\alpha)d\alpha , \tag{221}$$

where $m_2 = \log(r_{11}/r)$, $r_{11}^2 = x_o^2 + y_o^2$,

the quantities F_{11}, F_{22}, F_{33}, F_{44}, F_{55}, F_{66} are the same as given in Appendix II to chapter X .

Finally , the stresses in the wedge for matrix and inclusion are given by

$$\begin{aligned} (P_{rr})_m &= \{(P_{rr})_m\}_1 + (P_{rr})_2 , \\ (P_{\theta\theta})_m &= \{(P_{\theta\theta})_m\}_1 + (P_{\theta\theta})_2 , \\ (P_{r\theta})_m &= \{(P_{r\theta})_m\}_1 + (P_{r\theta})_2 \end{aligned} \tag{222}$$

and

$$\begin{aligned} (P_{rr})_i &= \{(P_{rr})_i\}_1 + (P_{rr})_2 , \\ (P_{\theta\theta})_i &= \{(P_{\theta\theta})_i\}_1 + (P_{\theta\theta})_2 , \\ (P_{r\theta})_i &= \{(P_{r\theta})_i\}_1 + (P_{r\theta})_2 , \end{aligned} \tag{223}$$

The expressions of $f_{11}(s, \theta_o)$, $f_{11}(s,-\theta_o)$, $f_{22}(s, \theta_o)$ and $f_{22}(s,-\theta_o)$ for this problem in real and imaginary parts are given below.

$$f_{11}(\alpha, \theta_o) = - \frac{2\mu\pi\, \epsilon_1 r'^2}{(K+1)\sin^3\theta_1 \cosh \pi\alpha} \Big[2 \sin(\theta_o-\beta_{11}) \Big\{$$

$$\frac{1}{2} \sin \frac{\theta_1}{2} \sin \theta_1 \cdot \cosh \alpha(\pi-\theta_1)$$

$$- \alpha \sin\theta_1 \cos \frac{\theta_1}{2} \sinh \alpha(\pi-\theta_1) - \cos \frac{\theta_1}{2} \cosh \alpha(\pi-\theta_1) \Big\}$$

$$+ \sin 2(\theta_o-\beta_{11}) \Big\{ \frac{1}{2} \sin \theta_1 \sin \frac{3\theta_1}{2} \cosh \alpha(\pi- \theta_1)$$

$$+ \alpha \sin \theta_1 \cos \frac{3\theta_1}{2} \sinh \alpha(\pi-\theta_1) + \cos \frac{\theta_1}{2} \cosh \alpha(\pi-\theta_1) \Big\} \Big]$$

$$- \frac{2i\mu\pi\, \epsilon_1 r'^2}{(K+1)\sin^3\theta_1 \cosh \pi\alpha} \Big[2 \sin(\theta_o-\beta_{11}) \Big\{ - \sin \frac{\theta_1}{2} \cdot$$

$$\sinh \alpha(\pi-\theta_1) + \alpha \sin \theta_1 \sin \frac{\theta_1}{2} \cosh \alpha(\pi-\theta_1)$$

$$+ \frac{1}{2} \sin \theta_1 \cos \frac{\theta_1}{2} \sinh \alpha(\pi- \theta_1) \Big\} + \sin 2(\theta_o-\beta_{11}) \cdot$$

$$\Big\{ - \sin \frac{\theta_1}{2} \sinh \alpha(\pi-\theta_1) - \alpha \sin \theta_1 \sin \frac{3\theta_1}{2} \cosh \alpha(\pi-\theta_1)$$

$$+ \frac{1}{2} \sin \theta_1 \cos \frac{3\theta_1}{2} \sinh \alpha(\pi-\theta_1) \Big\} \Big] \quad ,$$

where $$\cos \theta_1 = \frac{x_o \cos\theta_o + y_o \sin\theta_o}{(x_o^2 + y_o^2)^{\frac{1}{2}}} \quad ;$$

$f_{11}(\alpha, -\theta_o)$ may be obtained from the expression of $f_{11}(\alpha, \theta_o)$ by replacing θ_o by $-\theta_o$ and θ_1 by θ_2 where

$$\cos\theta_2 = \frac{x_o\cos\theta_o - y_o\sin\theta_o}{(x_o^2 + y_o^2)^{1/2}},$$

$$f_{22}(\alpha,\theta_o) = -\frac{2\pi\mu e_1 r'^2}{(K+1)\sin^3\theta_1\cosh\pi\alpha}\Big[\frac{3}{2}\sin\frac{\theta_1}{2}\cosh\alpha(\pi-\theta_1)\sin\theta_1$$

$$+\alpha\sin\theta_1\cos\frac{\theta_1}{2}\sinh\alpha(\pi-\theta_1) + \cos\frac{3\theta_1}{2}\cosh\alpha(\pi-\theta_1)$$

$$+2\cos(\theta_o-\beta_{11})\Big\{\frac{1}{2}\sin\frac{\theta_1}{2}\sin\theta_1\cosh\alpha(\pi-\theta_1)$$

$$-\alpha\sin\theta_1\cos\frac{\theta_1}{2}\sinh\alpha(\pi-\theta_1) - \cos\frac{\theta_1}{2}\cosh\alpha(\pi-\theta_1)\Big\}$$

$$+\cos 2(\theta_o-\beta_{11})\Big\{\frac{1}{2}\sin\theta_1\sin\frac{3\theta_1}{2}\cosh\alpha(\pi-\theta_1)$$

$$+\alpha\sin\theta_1\cos\frac{3\theta_1}{2}\sinh\alpha(\pi-\theta_1) + \cos\frac{\theta_1}{2}\cosh\alpha(\pi-\theta_1)\Big\}\Big]$$

$$-\frac{2i\pi\mu e_1 r'^2}{(K+1)\sin^3\theta_1\cosh\pi\alpha}\Big[\sin\frac{3\theta_1}{2}\sinh\alpha(\pi-\theta_1) + \alpha\sin\theta_1\cdot$$

$$\sin\frac{\theta_1}{2}\cosh\alpha(\pi-\theta_1) - \frac{3}{2}\sin\theta_1\cos\frac{\theta_1}{2}\sinh\alpha(\pi-\theta_1)$$

$$+2\cos(\theta_o-\beta_{11})\Big\{-\sin\frac{\theta_1}{2}\sinh\alpha(\pi-\theta_1) +$$

$$+ \alpha \sin\theta_1 \sin\frac{\theta_1}{2}\cosh\alpha(\pi-\theta_1) + \frac{1}{2}\sin\theta_1\cos\frac{\theta_1}{2}\cdot$$

$$\sinh\alpha(\pi-\theta_1) + \cos 2(\theta_o-\beta_{11}) - \sin\frac{\theta_1}{2}\sinh\alpha(\pi-\theta_1)$$

$$- \alpha\sin\theta_1\sin\frac{3\theta_1}{2}\cosh\alpha(\pi-\theta_1) + \frac{1}{2}\sin\theta_1\cos\frac{3\theta_1}{2}\cdot$$

$$\sinh\alpha(\pi-\theta_1) \qquad (224)$$

$f_{22}(\alpha_1-\theta_o)$ may be obtained from the expression of $f_{22}(\alpha, \theta_o)$ by replacing θ_o by $-\theta_o$ and θ_1 by θ_2 .

It may be verified from (214) and (221) that on the faces of the wedge $\theta = \pm\ \theta_o$, normal and shearing stresses vanish as they should. The continuity of normal and shearing stresses at the inclusion boundary may also be verified. The results of circular inclusion in a half plane may be obtained by taking the semi angle of the wedge as $\pi/2$.

BIBLIOGRAPHY

1. Green, A.E. and Zerna, W. 'Theoretical Elasticity' Oxford, (1960).

2. Muskhelishvili, N.I. 'Some Basic Problems of the Mathematical Theory of Elasticity' P. Noordhoff Ltd., Groningen, The Netherlands (1963).

3. Sokolnikoff, I.S. 'Mathematical Theory of Elasticity' McGraw-Hill Book Co., New York, (1956).

4. Timoshenko, S. and Goodier, J.N. 'Theory of Elasticity' McGraw-Hill Book Co., (1951).

5. Nabarro, F.R.N. Proc. Phys.Soc., 52 (1940) 90.

6. Bhargava, R.D. and Kapoor, O.P. Proc. Nat. Inst.Sci. India, A, 32 (1966) 418 .

7. Bhargava, R.D. and D. Pande Jour. of Science and Engg. Research, Vol. VII, Part 2, 1968.

8. Bhargava, R.D. and Radhakrishna, H.C. Proc. Camb. Phil. Soc., 59(1963) 811.

9. Bhargava, R.D. and Kapoor, O.P. Proc. Nat. Inst.Sci. India, A, 32 (1966) 46.

10. Bhargava, R.D. Appl. Sci. Res., A, 11(1961)80.

11. Eshelby, J.D. Proc. Roy.Soc., A, 241(1957)376.

12. Bhargava, R.D. and Radhakrishna, H.C. Proc. Camb.Phil.Soc., 59(1963)821.

13. Bhargava, R.D. and Radhakrishna, H.C. J. Phys. Soc. Japan, 19(1964)756.

14. Eshelby, J.D. Proc. Roy.Soc.,A, 152(1959)561.

15. Jaswon, M.A. and Bhargava, R.D. — Proc. Camb.Phil.Soc., 57(1961) 669.

16. Love, A.E.H. — 'A Treatise on the Mathematical Theory of Elasticity' Camb. University Press,(1959).

17. Sneddon, I.N. — 'Fourier Transforms' McGraw-Hill Book Co.,(1951).

18. Erdélyi, A. — 'Tables of Integral Transform' Volume I, McGraw-Hill Book Co., (1954).

19. Erdélyi, A. — 'Higher Transcendental Functions' Volume I, McGraw-Hill Book Co., (1953).

20. Hobson, E.W. — 'The Theory of Spherical and Ellipsoidal Harmonics' Chelsea Publishing Company, New York, (1955).

21. Kopal, Zdenek — 'Numerical Analysis' N.Y., Wiley, (1961).

22. Campbell, A. and Foster, R.M. — 'Fourier Integrals for Practical Applications', D. Van Nostrand Company, (1961).

23. Erdélyi, A. — 'Higher Transcendental Functions' Volume II, McGraw-Hill Book Co., (1953).

24. Jahnke, Emde and Lösch — 'Tables of Higher Functions' McGraw-Hill Book Co., (1960).

25. Frenkel, J. — 'Kinetic Theory of Liquids' Oxford (1946).

26. Mott, N.F. and Nabarro, F.R.N. — 'Proc. Phys. Soc., 52(1940)86.

27. Willis, J.R. — 'Quart. Journ. Mech. and Appl. Math.', 16 (1965) 157.

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